On the reductions of some crystalline representations

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Abstract. In this work we study a semilinear category $\mathrm{Mod}_k^{\mathrm{SD}}$ which appears as a full subcategory of the category of p-torsion Breuil–Kisin modules. We view $\mathrm{Mod}_k^{\mathrm{SD}}$ as extending Fontaine–Laffaille theory (for p-torsion coefficients) to weights contained in the range $[0, p]$. As an application we relate the restriction to inertia of a residual representation $\overline{\rho}$, with the weights $\subset [0, p]$ for which $\overline{\rho}$ has a crystalline lift. This allows us to deduce some new cases of the weight part of Serre's conjecture for unitary groups of rank n.

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CHAPTER 1

Introduction

Let p be a prime. We study a category described in terms of semilinear algebra, which by results of Gee–Liu–Savitt is closely related to the reduction modulo p of crystalline representations with Hodge–Tate weights in the interval $[0,p]$. While more complicated than the category of p-torsion Fontaine–Laffaille modules, we show these two categories behave sufficiently similarly that we are able to extend results on the reduction of crystalline representations with Hodge–Tate weights in $[0,p-1]$ to weights in $[0,p]$.

We begin this introduction by explaining the applications of our results to Galois representations and the weight part of Serre's conjecture. Then we give more details on the integral p-adic Hodge theory used to deduce these applications. In particular we describe the semilinear category alluded to above.

1. Inertial Weights

The weight part of Serre's conjecture, and its generalisations beyond the classical case of GL_2 over $\mathbb{Q}$, is closely related to the following local problem. For a continuous residual representation $\overline{\rho}$ of the Galois group of a local field $K/\mathbb{Q}_p$ determine the collection of Hodge–Tate weights for which $\overline{\rho}$ admits a crystalline lift. We refer to [**15**] for a detailed discussion of the relationship between these two problems.

In this work we assume $K/\mathbb{Q}_p$ is unramified and let k denote the residue field of K. To any Galois representation $\overline{\rho} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ one associates a set of *weights* $W(\overline{\rho})^{\mathrm{crys}}$. By a weight $\lambda = (\lambda_\tau)_{\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k,\overline{\mathbb{F}}_p)}$ we mean a collection of n-tuples of distinct integers λ_τ indexed by embeddings $\tau : k \to \overline{\mathbb{F}}_p$. We assert that $\lambda \in W(\overline{\rho})^{\mathrm{crys}}$ if and only if $\overline{\rho}$ has a crystalline lift (i.e. there exists a crystalline representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ such that $\overline{\rho} = \rho \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{F}}_p$) with τ-th Hodge–Tate weights λ_τ. Recent advances in modularity lifting results lead to another natural set of weights $W(\overline{\rho})^{\mathrm{diag}} \subset W(\overline{\rho})^{\mathrm{crys}}$, defined to be the subset containing those λ for which $\overline{\rho}$ admits a crystalline potentially diagonalisable lift (in the sense of [**2**]) with τ-th Hodge–Tate weight λ_τ. In this work we consider a third explicit set of weights $W(\overline{\rho})^{\mathrm{inert}}$ (a precise description is given below) which depends only on $\overline{\rho}^{\mathrm{ss}}|_{I_K}$ (see Lemma 7.3.4) which we compare with $W(\overline{\rho})^{\mathrm{crys}}$ and $W(\overline{\rho})^{\mathrm{diag}}$. If $\overline{\rho}$ is semisimple then standard arguments give that $W(\overline{\rho})^{\mathrm{inert}} \subset W(\overline{\rho})^{\mathrm{diag}}$.

For $* \in \{\text{inert}, \text{diag}, \text{crys}\}$ let $W(\overline{\rho})^*_{\leq p}$ denote the subset containing those $\lambda = (\lambda_\tau)$ such that $\lambda_\tau = (\lambda_{\tau,1} > \ldots > \lambda_{\tau,n})$ with $\lambda_{\tau,1} - \lambda_{\tau,n} \leq p$. We prove:

THEOREM A (Theorem 7.3.2). *Let $\overline{\rho}$ be as above. Then $W(\overline{\rho})^{\text{crys}}_{\leq p} \subset W(\overline{\rho})^{\text{inert}}_{\leq p}$. In particular, if $\overline{\rho}$ is semisimple then $W(\overline{\rho})^{\text{diag}}_{\leq p} = W(\overline{\rho})^{\text{crys}}_{\leq p} = W(\overline{\rho})^{\text{inert}}_{\leq p}$.*

In some cases we are able to prove that $W(\overline{\rho})^{\text{crys}}_{\leq p} = W(\overline{\rho})^{\text{diag}}_{\leq p}$ without the assumption that $\overline{\rho}$ is semisimple (Corollary 8.2.3). These results were already known when $n = 2$ by the work of Gee–Liu–Savit [**16**] and this thesis should be viewed as an extension of their methods to higher dimensions.

The relevance of this result is that, after the work of [**2**] and [**1**], producing potentially diagonalisable lifts of a given Hodge–Tate weight is a key obstruction to proving change of weight results for automorphic Galois representations. Combining Theorem A (and the non-semisimple cases mentioned above) with [**1**] we deduce the following (for any unfamiliar notation we refer to Chapter 10).

THEOREM B. *Let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, that every place F^+ dividing p splits completely in F, and that if n is even then $n[F : \mathbb{Q}]/2$ is even. Assume further that $p > 2$ is unramified in F. Suppose that $\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is an irreducible representation with split ramification. Assume that:*

- *There is an RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that*
  - *$\overline{r} \cong \overline{r}_{p,\iota}(\Pi)$.*
  - *For each place $w|p$ of F, $r_{p,\iota}(\Pi)|_{G_{F_w}}$ is potentially diagonalisable.*
  - *For each place $w|p$ of F, $\overline{r}|_{G_{F_w}}$ satisfies one of the following conditions*
    1. *is semisimple,*
    2. *is cyclotomic-free (see Notation 3.3.1) and a successive extension of characters,*
    3. *or $F_w = \mathbb{Q}_p$, $\overline{r}|_{G_{F_w}}$ is cyclotomic-free and each irreducible subquotient has dimension ≤ 4.*
  - *$\overline{r}(G_{F(\zeta_p)})$ is adequate.*

Let a be a global Serre weight and assume that[1] $a \in W(\overline{r})^{\text{crys}}_{\leq p}$. Then $\overline{r}$ is modular of weight a.

Let us say a word about the assumption that our weights differ by at most p. Ideally one would like a good understanding of the full set of weights $W(\overline{\rho})^{\text{crys}}$. However a given $\overline{\rho}$ will admit more crystalline lifts the further

[1]Note here $W(\overline{r})^{\text{crys}}_{\leq p}$ refers to a set of global Serre weights, rather than the local set of weights defined above.

apart one allows the Hodge–Tate weights to be, and this is reflected in the fact that the full set of weights $W(\overline{\rho})^{\text{crys}}$ is very complicated. In particular the behaviour witnessed by Theorem A does extend to weights further than p apart, and the inclusion $W(\overline{\rho})^{\text{inert}} \subset W(\overline{\rho})^{\text{crys}}$ will be very far from an equality. It is currently unknown to what extent one should expect $W(\overline{\rho})^{\text{diag}} = W(\overline{r})^{\text{crys}}$. In the context of the weight part of Serre's conjecture, a proof in dimension n will (or is likely to) require a good understanding of the set $W(\overline{\rho})^{\text{crys}}_{\leq (n-1)p}$ of weights with $\lambda_{\tau,1} - \lambda_{\tau,n} \leq (n-1)p$. For this reason Theorem B does not prove modularity of all expected weights (except when $n = 2$, and this was already known by [**16**]).

We conclude this part of the introduction by giving a precise description of $W(\overline{\rho})^{\text{inert}}$. Recall that the continuous irreducible representations of G_K on an $\overline{\mathbb{F}}_p$-vector space have a simple structure: each is induced from a character over an unramified extension of K. These characters can also be described explicitly. If L/K is an unramified extension with residue field l and $f_L = [l : \mathbb{F}_p]$ then any continuous character $G_L \to \overline{\mathbb{F}}_p^\times$ can be expressed as a product

$$\psi \prod \chi_\theta^{-r_\theta}$$

where ψ is an unramified character, where θ runs over the elements of $\mathrm{Hom}_{\mathbb{F}_p}(l, \overline{\mathbb{F}}_p)$, and where the χ_θ are the characters defined as follows. For $\theta \in \mathrm{Hom}_{\mathbb{F}_p}(l, \overline{\mathbb{F}}_p)$ the θ-th fundamental character is the composite

$$\chi_\theta : G_L \to \mathcal{O}_{L(\pi_L)}^\times \to l^\times \xrightarrow{\theta} \overline{\mathbb{F}}_p^\times$$

where π_L is any choice of $p^{f_L} - 1$-th root of a uniformiser of K and the first map is given by $\sigma \mapsto \sigma(\pi_L)/\pi_L$. Using the relation $\chi_\theta^p = \chi_{\theta \circ \varphi}$ one can write every character uniquely as above with the $r_\theta \in [0, p-1]$. In this work however it is important to allow r_θ to vary arbitrarily; thus to every character we associate a whole collection of multisets of integers. For integers $(r_\theta)_{\theta \in \mathrm{Hom}_{\mathbb{F}_p}(l, \overline{\mathbb{F}}_p)}$ define $\mathrm{Ind}_L(r_\theta) = \mathrm{Ind}_L^K(\prod_\theta \chi_\theta^{-r_\theta})$. While the χ_θ depend on the choice of uniformiser of K, they do so only up to unramified twist. Thus $\mathrm{Ind}_L(r_\theta)$ is well-defined up to unramified twist. We can express

$$\overline{\rho}^{\text{ss}} \cong \bigoplus \psi_L \otimes \mathrm{Ind}_L(r_\theta) \tag{1.1.1}$$

with the sum running over a collection (possibly with multiplicity) of unramified extensions L/K. We define $W(\overline{\rho})^{\text{inert}}$ to be the set of weights $\lambda = (\lambda_\tau)$ such that $\overline{\rho}^{\text{ss}}$ may be expressed as in (1.1.1) such that

$$\{\lambda_{\tau,1}, \dots, \lambda_{\tau,n}\} = \{r_\theta \mid \theta|_k = \tau\}$$

Note the $-r_\theta$ appearing in the definition of $\mathrm{Ind}_L(r_\theta)$; thus for Theorem A to be true we must normalize our Hodge–Tate weights such that the cyclotomic character has Hodge–Tate weight -1, and we maintain this convention throughout.

2. Semilinear Algebra

To prove Theorem A we define and study a category of objects, defined in terms of semilinear algebra, which is related to the reduction modulo p of crystalline representations with Hodge–Tate weights contained in the interval $[0,p]$.

In order to explain these semilinear objects we recall the notion of a Breuil–Kisin module. Fix a uniformiser of K and let $E(u)$ denote the minimal polynomial of that uniformiser over $K_0 = W(k)[\frac{1}{p}]$. A Breuil–Kisin module is a finitely generated module M over $\mathfrak{S} = W(k)[[u]]$ equipped with a semilinear (for the $\mathbb{Z}_p$-algebra endomorphism which on $W(k)$ lifts the p-th power map on k and sends $u \mapsto u^p$) endomorphism which becomes an isomorphism after inverting $E(u) \in \mathfrak{S}$. In particular we consider the following two collections of Breuil–Kisin modules:

- $\mathrm{Mod}^{\mathrm{BK}}_{\mathrm{free}}$, those Breuil–Kisin modules finite free over $\mathfrak{S}$.
- $\mathrm{Mod}^{\mathrm{BK}}_{k}$, those Breuil–Kisin modules finite free over $\mathfrak{S}/p = k[[u]]$.

(the notation in the second bullet point is used throughout this work, but the first is not).

The category $\mathrm{Mod}^{\mathrm{BK}}_{\mathrm{free}}$ appears in work of Breuil and Kisin on integral p-adic Hodge theory. Kisin [**20**] attaches to any crystalline $\rho : G_K \to \mathrm{GL}_n(\mathbb{Z}_p)$ a rank n object $M(\rho) \in \mathrm{Mod}^{\mathrm{BK}}_{\mathrm{free}}$ and proves that $\rho \mapsto M(\rho)$ is a fully faithful functor. The Breuil–Kisin module $M(\rho)$ can be viewed as an integral intermediary between the structures on the filtered isocrystal associated to $\rho[\frac{1}{p}]$ by Fontaine:

- There is a φ-equivariant identification of $M(\rho) \otimes_{\mathfrak{S}} W(k)$ with a $W(k)$-lattice inside $D_{\mathrm{crys}}(\rho[\frac{1}{p}])$. Here we view $W(k)$ as an $\mathfrak{S}$-algebra via $u \mapsto 0$.
- Viewing $\mathcal{O}_K$ as an $\mathfrak{S}$-algebra via $u \mapsto \pi$ (the fixed uniformiser of K) we can identify $M(\rho) \otimes_{\mathfrak{S}} \mathcal{O}_K$ with an $\mathcal{O}_K$-lattice inside $D_{\mathrm{dR}}(\rho[\frac{1}{p}])$. Moreover $M(\rho)$ (or more correctly the frobenius twist of $M(\rho)$) can be equipped with a filtration which, after inverting p, gives the filtration on $\mathrm{D}_{\mathrm{dR}}(\rho[\frac{1}{p}])$. A consequence is that the Hodge–Tate weights can be read off of $M(\rho)[\frac{1}{p}]$ as follows. Since $\mathfrak{S}[\frac{1}{p}]$ is a principal ideal domain the theory of elementary divisors applies and for any choice of $\mathfrak{S}$-basis $(e_1,\dots,e_n)$ of $M(\rho)$ we can write
$$\varphi(e_1,\dots,e_n) = (e_1,\dots,e_n)A\Lambda B \tag{1.2.1}$$
with $A,B \in \mathrm{GL}_n(\mathfrak{S}[\frac{1}{p}])$ and $\Lambda = \mathrm{diag}(E^{r_i})$. The r_i are then the Hodge–Tate weights of ρ. We emphasise the necessity of inverting p in this construction.
- Let $K_\infty = K(\pi^{1/p^\infty})$ be the field obtained by adjoining a compatible system of p-th power roots of π to K. Then, from $M(\rho)$ one can recover $\rho|_{G_{K_\infty}}$ via $\rho|_{G_{K_\infty}} = (M(\rho) \otimes_{\mathfrak{S}} W(C^\flat))^{\varphi=1}$. The ring $W(C^\flat)$ is one of Fontaine's period rings and the G_{K_∞}-action is obtained by

asserting that G_{K_∞} acts trivially on $M(\rho)$ and acts in the natural way on $W(C^\flat)$.

In fact the formula $M \mapsto T(M) = (M \otimes_{\mathfrak{S}} W(C^\flat))^{\varphi=1}$ makes sense for any Breuil–Kisin module M, and describes an exact functor from the category of Breuil–Kisin modules to the category of G_{K_∞}-representations. Exactness of this functor means that if $\overline{M}(\rho) = M(\rho) \otimes_{\mathfrak{S}} k[[u]] \in \mathrm{Mod}_k^{\mathrm{BK}}$ and $\overline{\rho} = \rho \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ then $T(\overline{M}(\rho)) = \overline{\rho}|_{G_{K_\infty}}$. The inertial weights $W(\overline{\rho})^{\mathrm{inert}}$ of $\overline{\rho}$ can be recovered from $\overline{\rho}|_{G_{K_\infty}}$, so the inertial weights of $\overline{\rho}$ can be recovered from $\overline{M}(\rho)$.

The starting point of this work is a theorem of Gee–Liu–Savitt [**16**] which says that if K is unramified over $\mathbb{Q}_p$ and the Hodge–Tate weights of ρ lie in the interval $[0, p]$ then the second bullet point above is valid without inverting p. This implies that the Hodge–Tate weights of ρ are observed by $\overline{M}(\rho)$. A precise statement of the result of Gee–Liu–Savitt is that if $K/\mathbb{Q}_p$ is unramified then there exists an $\mathfrak{S}$-basis of $M(\rho)$ such that (1.2.1) is valid with $A, B \in \mathrm{GL}_n(\mathfrak{S})$ and $B \equiv 1$ modulo p. This shows that with the following definition $\overline{M}(\rho) \in \mathrm{Mod}_k^{\mathrm{SD}}$ when the Hodge–Tate weights of ρ are contained in $[0, p]$.

Definition 1.2.2. We will study the full subcategory $\mathrm{Mod}_k^{\mathrm{SD}}$ inside $\mathrm{Mod}_k^{\mathrm{BK}}$ whose objects are those $\overline{M} \in \mathrm{Mod}_k^{\mathrm{BK}}$ satisfying the following two conditions.

(1) There exists a $k[[u]]$-basis $(e_1, \dots, e_n)$ such that
$$\varphi(e_1, \dots, e_n) = (e_1, \dots, e_n) A\Lambda$$
where $A \in \mathrm{GL}_n(k[[u]])$ and $\Lambda = \mathrm{diag}(u^{r_i})$.

(2) The integers r_i lies in the interval $[0, p]$. We call the integers r_i the weights of $\overline{M}$.

For the rest of this introduction assume K is unramified over $\mathbb{Q}_p$.

Remark 1.2.3. Since $k[[u]]$ is a principal ideal domain, if $\overline{M} \in \mathrm{Mod}_k^{\mathrm{BK}}$ then for any choice of $k[[u]]$-basis of $\overline{M}$ we can write $\varphi = A\Lambda B$ with $A, B \in \mathrm{GL}_n(k[[u]])$ and Λ diagonal with powers of u along the diagonal. Thus we make sense of the weights of $\overline{M}$, they are the r_i such that $\Lambda = \mathrm{diag}(u^{r_i})$. Being in $\mathrm{Mod}_k^{\mathrm{SD}}$ means that weights of $\overline{M}$ lie in the interval $[0, p]$ and that $B \in \mathrm{GL}_n(k[[u^p]])$.

Note that the representations considered in the previous section were all valued in $\overline{\mathbb{Z}}_p$ or $\overline{\mathbb{F}}_p$; this is necessary to even make sense of inertial weights of a residual representation. On the other hand Breuil–Kisin modules only behave well with respect to coefficients which are finite over $\mathbb{Z}_p$. To recover the setting of the first section we therefore fix a finite extension $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k_E. We study a variant $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ of $\mathrm{Mod}_k^{\mathrm{BK}}$ whose objects consist of $M \in \mathrm{Mod}_k^{\mathrm{BK}}$ equipped with a $\mathcal{O}$-action (a homomorphism $\mathcal{O} \to \mathrm{End}_{\mathrm{BK}}(M)$). Likewise we make sense of $\mathrm{Mod}_{\mathrm{free}}^{\mathrm{BK}}(\mathcal{O})$

and $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$. For any Breuil–Kisin module M with such an $\mathcal{O}$-action, the functorality of $M \mapsto T(M)$ means that $T(M)$ is an $\mathcal{O}$-valued representation.

Throughout we allow ourselves to take E as large as we like so as to avoid any rationality issues. This allows us, for any $\overline{M} \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$, to make sense of the inertial weights of the k_E-valued representation $T(M)$ (note the discussion of inertial weights from the previous section is made for representations of G_K, but residual representations of G_{K_∞} behave in an entirely analogous manner).

Theorem C. *Let $\overline{N} \in \mathrm{Mod}_k^{\mathrm{SD}}$.*

(1) *Let $0 \to \overline{M} \to \overline{N} \to \overline{P} \to 0$ be an exact sequence in $\mathrm{Mod}_k^{\mathrm{BK}}$. Then $\overline{M}$ and $\overline{P}$ are in $\mathrm{Mod}_k^{\mathrm{SD}}$ and the weights of $\overline{N}$ equal the union of the weights of $\overline{M}$ and $\overline{P}$.*
(2) *Suppose $\overline{N}$ is irreducible as an object of $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ (this is the same as $T(\overline{N})$ being irreducible as a k_E-representation of G_{K_∞}) and has distinct weights. Then the weights of $\overline{N}$ coincide with the inertial weights of $T(\overline{N})$.*

See Proposition 4.3.13 for (1) and Corollary 7.3.1 for (2). Using the result of Gee–Liu–Savitt mentioned above, it is straightforward to deduce Theorem A from this result. As well as Theorem C we are able to establish some additional results concerning the categories $\mathrm{Mod}_k^{\mathrm{SD}}$ and $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$. The following theorem motivates our use of the term *strongly divisible* (see Lemma 6.1.2 and Proposition 6.2.2).

Theorem D. *Let $\overline{M} \in \mathrm{Mod}_k^{\mathrm{BK}}$ and assume that the weights of $\overline{M}$ are contained in the interval $[0, p-1]$.*

(1) *For such $\overline{M}$ there exists a functorial association $\overline{M} \mapsto \overline{M}_k$ into the category of p-torsion Fontaine–Laffialle modules. Further, $\overline{M} \in \mathrm{Mod}_k^{\mathrm{SD}}$ if and only if $\overline{M}_k$ is strongly divisible, in the sense of* [**12**].
(2) *If $\overline{M} \in \mathrm{Mod}_k^{\mathrm{SD}}$ and has weights contained in $[0, p-2]$ then there is a natural identification of (the dual of) $T(M)$ with the restriction to G_{K_∞} of the G_K-representation associated to $\overline{M}_k$ in* [**12**].

We see no reason that part (2) does not hold for weights in $[0, p-1]$ but we are unable to prove this. This theorem illustrates some relation between $\mathrm{Mod}_k^{\mathrm{SD}}$ and strongly divisible p-torsion Fontaine–Laffaille modules. Each of these latter objects arise from the reduction modulo p of some crystalline representation with Hodge–Tate weights in the interval $[0, p-1]$, and so we ask whether every $\overline{M} \in \mathrm{Mod}_k^{\mathrm{SD}}$ arises as $\overline{M}(\rho)$ for some crystalline $\rho : G_K \to \mathrm{GL}_n(\mathbb{Z}_p)$ with Hodge–Tate weights in $[0, p]$? Under some conditions we are able to answer this question in the affirmative.

Theorem E. *Let $\overline{M} \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$. Then $\overline{M} = \overline{M}(\rho)$ for some crystalline ρ with Hodge–Tate weights $[0, p]$ if $T(M)$ is cyclotomic-free (see Notation 3.3.1) and one of the following conditions is satisfied.*

(1) *The weights of $\overline{M}$ are contained in $[0, p-1]$.*

(2) *Each irreducible subquotient of* $T(\overline{M})$ *is a character.*
(3) $K = \mathbb{Q}_p$ *and each irreducible subquotient has dimension* ≤ 4.

The notion of cyclotomic-freeness is an n-dimensional extension of the avoidance of representations of the form $\left(\begin{smallmatrix} \chi_{\mathrm{cyc}} & * \\ 0 & 1 \end{smallmatrix}\right)$ where χ_{cyc} denotes the cyclotomic character.

The proof of this theorem goes in the same way that one typically produces crystalline lifts of residual representations, namely by induction on the length. One produces crystalline lifts of the irreducible subquotients of $\overline{M}$ (i.e. $\overline{M}$ with $T(\overline{M})$ irreducible as a G_{K_∞}-representation) and one then produces crystalline lifts of extensions. We do the latter by computing dimensions of ext groups in $\mathrm{Mod}_k^{\mathrm{SD}}$ and relating this dimension with the dimension of the space of crystalline extensions (in characteristic zero). It is in this step that the cyclotomic-freeness assumption is used. One is left needing to lift irreducible objects of $\mathrm{Mod}_k^{\mathrm{SD}}$. This is straightforward when the irreducible objects are of rank one (so the theorem follows in case (2)). When the weights of $\overline{M}$ are contained in $[0, p-1]$ we show $\overline{M}$ is *induced* from a rank one Breuil–Kisin module (as is the case in Fontaine–Laffaille theory) from which case (1) of the theorem follows. For low-dimensional reasons the same is true when in case (3). Unfortunately not every irreducible object of $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ is induced from a rank one object (and so in this respect $\mathrm{Mod}_k^{\mathrm{SD}}$ is more complicated than the category of p-torsion strongly divisible Fontaine–Laffaille theory) and we do not know how to find crystalline liftings in these cases.

If every $\overline{M} \in \mathrm{Mod}_k^{\mathrm{SD}}$ was of the form $\overline{M}(\rho)$ for some crystalline ρ then it would follow that the G_{K_∞}-representation $T(\overline{M})$ would extend to a G_K-representation. In fact one would know more. Recall the ring $\mathcal{O}_{C^\flat}$: as a ring this can be identified with the ring of integers of an algebraic closure of $k[[u]]$ but it comes with extra structure; in particular it admits a natural action of G_K. The most important ingredient of the result of Gee–Liu–Savitt [**16**] quoted above is that if ρ is crystalline then they show that $\overline{M}(\rho) \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ admits a semilinear G_K-action satisfying

$$(\sigma - 1)(m) \in \overline{M}(\rho) \otimes_{k[[u]]} u^{p/p-1}\mathcal{O}_{C^\flat}$$

for each $\sigma \in G_K$ and each $m \in \overline{M}(\rho)$. This is true without any assumption on the Hodge–Tate weights of ρ. We show:

Theorem F. *Let* $\overline{M} \in \mathrm{Mod}_k^{\mathrm{BK}}$ *with weights in the interval* $[0, p]$. *Assume also that* $\overline{M}$ *is* χ_{cyc}^p*-free (Definition 5.1.1). Then* $\overline{M} \in \mathrm{Mod}_k^{\mathrm{SD}}$ *if and only if* $\overline{M} \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ *admits a continuous* φ*-equivariant* G_K*-action satisfying*

$$(\sigma - 1)(m) \in \overline{M} \otimes_{k[[u]]} u^{p/p-1}\mathcal{O}_{C^\flat}$$

for $\sigma \in G_K$ *and* $m \in \overline{M}$.

The notation of being χ_{cyc}^p-free is somewhat similar condition to the cyclotomic-freeness assumption appearing in Theorem E. It is an n-dimensional version of the exclusion of Breuil–Kisin modules $\overline{M} \in \mathrm{Mod}_k^{\mathrm{BK}}$ which are extensions of the form $0 \to \overline{M}(\mathbb{Z}_p) \to \overline{M} \to \overline{M}(\mathbb{Z}_p(-p)) \to 0$.

3. Non-regular Hodge–Tate weights

The final part of this work involves an attempt to extend the range of Hodge–Tate weights beyond $[0, p]$. In general the above results are simply false outside this range, but we are able to show it can be done for Hodge–Tate weights which are sufficiently non-regular (we say Hodge–Tate weights are regular if they are all distinct; in particular the Hodge–Tate weights discussed in the first section are regular). The next result indicates what we can prove for representations which are sometimes called F-crystalline. In practice we are able to be more flexible about the kind of non-regular Hodge–Tate weights we consider (see Theorem 9.4.14) but we restrict to the F-crystalline case to illustrate ideas.

THEOREM G. *Fix an embedding $\tau : k \to \overline{\mathbb{F}}_p$ and let $f = [K : \mathbb{Q}_p]$. Let $\rho : G_K \to \mathrm{GL}_n(\mathcal{O})$ be a crystalline representation and assume that for $\tau \neq \tau'$ the τ'-th Hodge–Tate weights of ρ are all zero. Assume further that the τ-th Hodge–Tate weights are contained in the interval*

$$[0, p^f - x_f]$$

where $x_f > 0$ is the smallest integer such that $f + x_f > v_p((p^f - x_f - 1)!)$. Then $\overline{M}(\rho)$ is in the full subcategory of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ whose objects satisfy condition (1) of Definition 1.2.2 and the weights of $\overline{M}(\rho)$ coincide with the Hodge–Tate weights of ρ.

Using the inequality $v_p(x!) \leq x/(p-1)$ we see that $x_f \leq p^{f-1} - f + \lceil \frac{f-1}{p} \rceil$. As a consequence of this theorem it is possible to replace all the results in this introduction with the interval $[0, p]$ replaced with the Hodge–Tate weights as described in the theorem, but we shall not spell this out in this work.

4. Previous Work

Questions similar to those answered by Theorem A have been considered by a number of people. When $n = 2$ and $K = \mathbb{Q}_p$ the result is a consequence of calculations due to Breuil (as described in [**3**, Théorème 3.2.1]). When $n = 2$, $p > 2$ and K is any unramified extension of $\mathbb{Q}_p$ then Theorem A is due to Gee–Liu–Savitt [**16**]. Our methods should be viewed as a generalisation to GL_n of their methods. In particular they consider the category $\mathrm{Mod}_k^{\mathrm{SD}}$ (though they do not give it this name) and prove part (1) of Theorem C for extensions of rank one Breuil–Kisin modules. These results were extended to allow $p = 2$ in the work of Wang [**29**].

There have also previously been extensions of the methods of Gee–Liu–Savitt beyond two dimensions. Gao [**13**] [**14**] has considered cases in which the residual representation is a successive extension of characters, and obtains results similar to part (2) of Theorem E, and Theorem B when all the local residual representations satisfy condition (2).

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CHAPTER 2

Overview of Crystalline Representations

In this chapter we give a brief introduction to the theory of crystalline representations. Additionally we gather together the tools from p-adic Hodge theory which shall be used in the following chapters to study these representations.

1. Period Rings

Throughout we let k be a finite field. We shall write K_0 for $W(k)[\frac{1}{p}]$. We let K denote a totally ramified extension of K_0 of degree e. Let C denote the completion of an algebraic closure $\overline{K}$ of K and let $\mathcal{O}_C$ be its ring of integers. Let us also fix a $\mathbb{Z}_p$-generator $\epsilon = (\epsilon_1, \epsilon_2, \dots)$ of $\mathbb{Z}_p(1)$. Thus $\epsilon_n \in \overline{K}$ satisfies $\epsilon_n^p = \epsilon_{n-1}$ and $\epsilon_1 = 1$.

Construction 2.1.1. Recall the construction of the *tilt* $\mathcal{O}_{C^\flat}$ of $\mathcal{O}_C$. As a ring it is defined to be the inverse limit $\varprojlim \mathcal{O}_C/p$ with transition maps given by $x \mapsto x^p$. This is a domain and we write $C^\flat$ for its fraction field. The obvious map $\varprojlim \mathcal{O}_C \to \mathcal{O}_{C^\flat}$ (again with transition maps $x \mapsto x^p$) is a multiplicative bijection and the projection $\mathcal{O}_{C^\flat} \to \mathcal{O}_C$ onto the first coordinate yields a multiplicative map which we express as $x \mapsto x^\sharp$. Letting v_p denote the valuation on C normalised so that $v_p(p) = 1$ we obtain a valuation $v^\flat(x) = v_p(x^\sharp)$ on $C^\flat$ for which $C^\flat$ is complete. If $\pi^\flat \in \mathcal{O}_{C^\flat}$ is any element with $v^\flat(\pi^\flat) > 0$ then the $\pi^\flat$-adic topology on $\mathcal{O}_{C^\flat}$ coincides with the topology induced by $v^\flat$ and $C^\flat = \mathcal{O}_{C^\flat}[\frac{1}{\pi^\flat}]$.

Set $A_{\text{inf}} = W(\mathcal{O}_{C^\flat})$. There is a unique continuous homomorphism $\theta : A_{\text{inf}} \to \mathcal{O}_C$ sending $[x] \mapsto x^\sharp$ where $[\cdot]$ denotes the teichmuller map. Inside A_{inf} there are the following distinguished elements.

(1) $\mu = [\epsilon] - 1$
(2) $\xi = \frac{\mu}{\varphi^{-1}(\mu)}$

The kernel of θ is a principal ideal and a necessary and sufficient condition that $x \in \ker\theta$ is a generator is that $v^\flat(\overline{x}) = 1$ where $\overline{x} \in \mathcal{O}_{C^\flat}$ denotes the reduction modulo p of x. Thus ξ generates the kernel of θ. Let B_{dR}^+ be the ξ-adic completion of $A_{\text{inf}}[1/p]$. It is a complete discrete valuation ring with residue field C. There is a homomorphism $\mathbb{Z}_p(1) \to B_{\text{dR}}^+$ which sends a system $\zeta = (\zeta_1, \zeta_2, \ldots) \in \mathbb{Z}_p(1)$ onto the element

$$\log([\zeta]) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\zeta]-1)^n}{n} \in B_{\mathrm{dR}}^+$$

We set $t = \log([\epsilon])$, which is a uniformizer of B_{dR}^+. The action of $G_K = \mathrm{Gal}(\overline{K}/K)$ on $\mathcal{O}_C$ induces an action on $\mathcal{O}_{C^\flat}$ and hence on A_{inf}. Since the ideal (ξ) is stable under this action G_K also acts on B_{dR}^+. The element t satisfies $\sigma(t) = \chi_{\mathrm{cyc}}(\sigma)t$ for all $\sigma \in G_K = \mathrm{Gal}(\overline{K}/K)$, where χ_{cyc} denotes the cyclotomic character.

Construction 2.1.2. Let $\nu \in A_{\mathrm{inf}}$ be a generator of $\ker\theta$. Let $A_{\max}$ be the subring of B_{dR}^+ consisting of those elements which can be written as $\sum_{n\geq 0} x_n (\frac{\nu}{p})^n$ with $x_n \in A_{\mathrm{inf}}$ a sequence which converges p-adically to zero in A_{inf}. This ring is p-adically complete and stable under the action of G_K. The automorphism φ on A_{inf} lifting $x \mapsto x^p$ on $\mathcal{O}_{C^\flat}$ extends to an injective $\mathbb{Z}_p$-algebra endomorphism of $A_{\max}$. The ring $A_{\max}$ does not depend on the choice of ν. We let $B_{\max}^+ = A_{\max}[\frac{1}{p}]$ and $B_{\max} = B_{\max}^+[\frac{1}{t}]$.

These rings are discussed in detail in [**6**]. As in *loc. cit.* we use the rings $A_{\max}, B_{\max}^+$ and $B_{\max}$ in place of the more usual period rings $A_{\mathrm{crys}}, B_{\mathrm{crys}}^+$ and B_{crys}. One advantage is that $B_{\max}^+$ is topologically better behaved than B_{crys}^+, as the following lemma indicates.

Lemma 2.1.3. *Equip $B_{\max}^+$ with the topology described by asserting that $(p^n A_{\max})_{n\geq 0}$ forms a basis of open neighbourhoods of 0. Then $B_{\max}^+$ is complete and any principal ideal $aB_{\max}^+ \subset B_{\max}^+$ is closed.*

Proof. Completeness of $B_{\max}^+$ is clear since $A_{\max}$ is p-adically complete. To check that a principal ideal $aB_{\max}^+$ is closed consider a sequence $b_i \in aB_{\max}^+$ which converges to $b \in B_{\max}^+$; we must show $b \in aB_{\max}^+$. Since $B_{\max}^+$ is a domain and complete it suffices to show $\frac{b_i}{a}$ converges in $B_{\max}^+$. That this is the case follows from [**6**, Proposition III.2.1] which asserts that if $||x|| = \inf_{\{n | x \in p^n A_{\max}\}} p^n$ then

$$p^{-1}||x||||y|| \leq ||xy|| \leq ||x||||y||$$

for all $x, y \in B_{\max}^+$ (the right inequality is obvious but the left inequality is not; in particular it is not true if $A_{\max}$ and $B_{\max}^+$ are replaced with A_{crys} and B_{crys}^+). This shows that

$$||\tfrac{b_i}{a} - \tfrac{b_j}{a}|| \leq \tfrac{p}{||a||}||b_i - b_j||$$

and so, as (b_i) is a convergent sequence, we conclude that $(\frac{b_i}{a})$ is Cauchy, and hence convergent. $\square$

2. Crystalline Representations

By a p-adic Galois representation we mean a finite dimensional $\mathbb{Q}_p$-vector space V equipped with a continuous $\mathbb{Q}_p$-linear action of the Galois group G_K.

In the following we shall make a number of assertions. These are all well-known when the ring $B_{\max}$ is replaced by the usual ring of crystalline periods B_{crys} (see for instance [**11**, Section 5]). From these standard results one easily deduces the statements we make here using that there are inclusions $\varphi(B_{\max}) \subset B_{\text{crys}} \subset B_{\max}$ (for this see [**6**, III.2]).

Construction 2.2.1. If V is a p-adic Galois representation then G_K acts on the tensor product $V \otimes_{\mathbb{Q}_p} B_{\max}$ via the respective actions on each component. Since $B_{\max}^{G_K} = K_0$ the G_K-invariants

$$D_{\text{crys}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\max})^{G_K}$$

form a K_0-vector space. It can be shown that multiplication $D_{\text{crys}}(V) \otimes_{K_0} B_{\max} \to V \otimes_{\mathbb{Q}_p} B_{\max}$ is injective which implies $D_{\text{crys}}(V)$ is finite-dimensional over K_0.

The injective frobenius endomorphism φ on $B_{\max}$ induces an injective (and therefore bijective) frobenius semilinear endomorphism φ of $D_{\text{crys}}(V)$.

Definition 2.2.2. A p-adic Galois representation V is crystalline if the natural φ, G_K-equivariant map

$$D_{\text{crys}}(V) \otimes_{K_0} B_{\max} \to V \otimes_{\mathbb{Q}_p} B_{\max} \tag{2.2.3}$$

is an isomorphism. This is equivalent to asking that $\dim_{K_0} D_{\text{crys}}(V) = \dim_{\mathbb{Q}_p} V$.

Remark 2.2.4. Any crystalline representation is de Rham in the sense that the inclusion $B_{\max} \otimes_{K_0} K \subset B_{\text{dR}}$ induces an equality between $D_{\text{crys}}(V)_K := D_{\text{crys}}(V) \otimes_{K_0} K$ and $(V \otimes_{\mathbb{Q}_p} B_{\text{dR}})^{G_K}$. The filtration $F^i B_{\text{dR}} = t^i B_{\text{dR}}^+$ induces a filtration $F^i D_{\text{crys}}(V)_K = (V \otimes_{\mathbb{Q}_p} t^i B_{\text{dR}}^+)^{G_K}$ on $D_{\text{crys}}(V)_K$ such that tensoring (2.2.3) up to B_{dR} gives a G_K-equivariant identification

$$D_{\text{crys}}(V) \otimes_{K_0} B_{\text{dR}} = V \otimes_{\mathbb{Q}_p} B_{\text{dR}}$$

of filtered modules (where the filtrations on the tensor products are given by $F^n = \sum_{i+j=n} F^i \otimes F^j$, and V is given the trivial filtration $F^0 V = V, F^{-1} V = 0$).

Fontaine [**11**, Théorème 5.3.5] has proven the following result.

Proposition 2.2.5. *The functor $V \mapsto D_{\text{crys}}(V)$ from the category of crystalline representations into the category of filtered φ-modules (i.e finite dimensional K_0-vector spaces D equipped with a frobenius semilinear automorphism and a filtration on D_K) is a fully faithful, exact $\otimes$-functor*[1].

[1] By a $\otimes$-functor we mean a functor which is compatible with tensor products and duals defined suitably in either category.

DEFINITION 2.2.6. If V is a crystalline representation $\mathrm{HT}(V)$, the Hodge–Tate weights of V, is the multiset of integers which contains i with multiplicity equal to

$$\dim_K \mathrm{gr}^i(D_{\mathrm{crys}}(V)_K)$$

Thus the cyclotomic character has Hodge–Tate weight -1.

3. Extensions of Crystalline Representations

If V is a p-adic Galois representation we let $H^i(G_K, V)$ denote continuous group cohomology. We let $\mathrm{Ext}^i(-,-)$ denote the i-th Yoneda extension group in the abelian category of p-adic Galois representations. Each of these groups are $\mathbb{Q}_p$-vector spaces.

CONSTRUCTION 2.3.1. Let V_i be a pair of p-adic Galois representations. To any class in $H^1(G_K, \mathrm{Hom}(V_2, V_1))$, represented by a 1-cocycle $c : G_K \to \mathrm{Hom}(V_2, V_1)$, one attaches the p-adic Galois representation whose underlying $\mathbb{Q}_p$-vector space is $V = V_1 \oplus V_2$ with G_K-action given by

$$\sigma((v_1, v_2)) = (\sigma(v_1) + c_\sigma \circ \sigma(v_2), \sigma(v_2))$$

(here we write c_σ in place of $c(\sigma)$ to ease notation). The representation V sits in an exact sequence $0 \to V_1 \to V \to V_2 \to 0$. This construction induces a homomorphism $H^1(G_K, \mathrm{Hom}(V_2, V_1)) \to \mathrm{Ext}^1(V_2, V_1)$.

Conversely, if $0 \to V_1 \to V \to V_2 \to 0$ is an exact sequence of p-adic Galois representations then any choice of splitting (in the category of $\mathbb{Q}_p$-vector spaces) gives rise to a 1-cocycle

$$c_\sigma : V_2 \xrightarrow{\sigma^{-1}} V_2 \to V \xrightarrow{\sigma} V \to V_1$$

This construction induces a homomorphism $\mathrm{Ext}^1(V_2, V_1) \to H^1(G_K, \mathrm{Hom}(V_2, V_1))$ which is inverse to the map of the previous paragraph.

DEFINITION 2.3.2. Define $\mathrm{Ext}^1_{\mathrm{crys}}(V_2, V_1) \subset \mathrm{Ext}^1(V_2, V_1)$ to be the subset whose elements are classes represented by exact sequences $0 \to V_1 \to V \to V_2 \to 0$ with V crystalline.

PROPOSITION 2.3.3. *Let V_i be a pair of crystalline p-adic Galois representations. Then $\mathrm{Ext}^1_{\mathrm{crys}}(V_2, V_1)$ is a subspace of $\mathrm{Ext}^1(V_2, V_1)$ of $\mathbb{Q}_p$-dimension*

$$[K : \mathbb{Q}_p]\,\mathrm{Card}(\{i - j < 0 \mid i \in \mathrm{HT}(V_1), j \in \mathrm{HT}(V_2)\}) + \dim_{\mathbb{Q}_p} \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V_2, V_1)$$

PROOF. Set $W = \mathrm{Hom}(V_2, V_1)$ and let $H^1_f(G_K, W)$ be the kernel of the map $H^1(G_K, W) \to H^1(G_K, W \otimes_{\mathbb{Q}_p} B_{\max})$. We claim the identification $H^1(G_K, W) \to \mathrm{Ext}^1(V_2, V_1)$ discussed in Construction 2.3.1 identifies $H^1_f(G_K, W)$ with $\mathrm{Ext}^1_{\mathrm{crys}}(V_2, V_1)$. This will imply that $\mathrm{Ext}^1_{\mathrm{crys}}(V_2, V_1)$ is a $\mathbb{Q}_p$-subspace.

To see the claim let c be a 1-cocyle representing a class in $H^1(G_K, W)$ and let $V = V_1 \oplus V_2$, with G_K-action given as in Construction 2.3.1, be the corresponding extension. It is immediate from the construction of V that the class of c is in $H^1_f(G_K, W)$ if and only if there exists a G_K-equivariant identification $V \otimes_{\mathbb{Q}_p} B_{\max} = (V_1 \otimes_{\mathbb{Q}_p} B_{\max}) \bigoplus (V_2 \otimes_{\mathbb{Q}_p} B_{\max})$.

Thus to prove $H^1_f(G_K, W)$ and $\mathrm{Ext}^1_{\mathrm{crys}}(V_2, V_1)$ are identified we just have to show that for V sitting in an exact sequence $0 \to V_1 \to V \to V_2 \to 0$ with V_i crystalline, V is crystalline if and only if this sequence splits G_K-equivariantly after tensoring with $B_{\max}$. If V is crystalline then, recalling that $D_{\mathrm{crys}}(-)$ is an exact functor, any $\mathbb{Q}_p$-splitting of $0 \to D_{\mathrm{crys}}(V_1) \to D_{\mathrm{crys}}(V) \to D_{\mathrm{crys}}(V_2) \to 0$ gives rise to a G_K-equivariant splitting of $0 \to V_1 \otimes_{\mathbb{Q}_p} B_{\max} \to V \otimes_{\mathbb{Q}_p} B_{\max} \to V_2 \otimes_{\mathbb{Q}_p} B_{\max} \to 0$. If instead we have such a G_K-equivariant splitting then the sequence stays exact after taking G_K-invariants and so we get an exact sequence

$$0 \to D_{\mathrm{crys}}(V_1) \to D_{\mathrm{crys}}(V) \to D_{\mathrm{crys}}(V_2) \to 0$$

We conclude that $\dim_{K_0} D_{\mathrm{crys}}(V) = \dim_{\mathbb{Q}_p} V_1 + \dim_{\mathbb{Q}_p} V_2 = \dim_{\mathbb{Q}_p} V$, and so V is crystalline. Thus we see that $H^1_f(G_K, W)$ and $\mathrm{Ext}^1_{\mathrm{crys}}(V_2, V_1)$ may be identified, as was claimed.

We remark that $\mathrm{HT}(\mathrm{Hom}(V_2, V_1)) = \{i - j \mid i \in \mathrm{HT}(V_1), j \in \mathrm{HT}(V_2)$. To see this we use that $V \mapsto D_{\mathrm{crys}}(V)$ is a $\otimes$-functor and so $D_{\mathrm{crys}}(\mathrm{Hom}(V_2, V_1)) = D_{\mathrm{crys}}(V_2)^\vee \otimes D_{\mathrm{crys}}(V_1)$. Now to complete the proof one can invoke the dimension formula for the H^1_f as described in e.g. [**23**, Proposition 1.24] (note that in *loc. cit.* the H^1_f whose dimension formula is computed is defined using B_{crys} in place of $B_{\max}$; it is however easy to see each description gives identical subspaces of H^1, using that $\varphi(B_{\max}) \subset B_{\mathrm{crys}} \subset B_{\max}$). ☐

4. Breuil–Kisin Modules

By a crystalline $\mathbb{Z}_p$-lattice we shall mean a G_K-stable $\mathbb{Z}_p$-lattice inside a crystalline p-adic Galois representation. All the constructions described in this section depend upon the choice of a uniformizer $\pi \in K$ and a compatible system of p-th power roots π^{1/p^n} of π. Such a compatible system is the same as an element $\pi^\flat = (\pi, \pi^{1/p}, \dots) \in \mathcal{O}_{C^\flat}$ with $(\pi^\flat)^\sharp = \pi$. Let us once and for all fix such a π and $\pi^\flat$.

We write K_∞ for $K(\pi^{1/p^\infty})$.

Notation 2.4.1. Let $\mathfrak{S} = W(k)[[u]]$. We equip this ring with a $\mathbb{Z}_p$-algebra endomorphism φ which acts on $W(k)$ as the frobenius and which sends $u \mapsto u^p$. View $\mathfrak{S}$ as a subring of A_{inf} via $\sum a_i u^i \mapsto \sum a_i [\pi^\flat]^i$. This embedding is compatible with the φ's on each ring. Let $E(u) \in \mathfrak{S}$ denote the Eisenstein polynomial over K_0 such that $E(\pi) = 0$.

Definition 2.4.2. A Breuil–Kisin module is a finitely generated $\mathfrak{S}$-module M equipped with an isomorphism[2]

$$\varphi_M : M \otimes_{\varphi, \mathfrak{S}} \mathfrak{S}[\tfrac{1}{E}] \to M[\tfrac{1}{E}]$$

[2]For $m \in M$ we write $\varphi_M(m)$ in place of $\varphi_M(m \otimes 1)$. Then φ_M describes a φ-semilinear map $M \to M[\frac{1}{E}]$. From this semilinear map $M \to M[\frac{1}{E}]$ one recovers $M \otimes_\varphi \mathfrak{S}[\frac{1}{E}] \to M[\frac{1}{E}]$ via $m \otimes \frac{s}{E^n} \mapsto \varphi(m) \frac{s}{E^n}$.

We denote the category of Breuil–Kisin modules by $\mathrm{Mod}_K^{\mathrm{BK}}$. We write M^φ for the image of $M \otimes_{\varphi,\mathfrak{S}} \mathfrak{S}$ under φ_M. When there is no risk of confusion we write φ for the isomorphism φ_M.

REMARK 2.4.3. The category $\mathrm{Mod}_K^{\mathrm{BK}}$ is an abelian category. Moreover this category admits natural notions of tensor product and internal hom:

- If M and N are two Breuil–Kisin modules then we write $M \otimes N$ for the Breuil–Kisin module with underlying $\mathfrak{S}$-module $M \otimes_{\mathfrak{S}} N$ and with frobenius given by $\varphi_M \otimes \varphi_N$.
- The set of $\mathfrak{S}$-linear homomorphisms $\mathrm{Hom}_{\mathfrak{S}}(M, N)$ between two Breuil–Kisin modules is made into a Breuil–Kisin module, written $\mathrm{Hom}(M, N)$, as we now explain. Since φ is flat on $\mathfrak{S}$ the natural map $\mathrm{Hom}_{\mathfrak{S}}(M,N) \otimes_\varphi \mathfrak{S}[\frac{1}{E}] \to \mathrm{Hom}_{\mathfrak{S}[\frac{1}{E}]}(M \otimes_\varphi \mathfrak{S}[\frac{1}{E}], N \otimes_\varphi \mathfrak{S}[\frac{1}{E}])$ is an isomorphism [**27**, Tag 087R]. Similarly the natural map $\mathrm{Hom}_{\mathfrak{S}}(M,N)[\frac{1}{E}] \to \mathrm{Hom}_{\mathfrak{S}[\frac{1}{E}]}(M[\frac{1}{E}], N[\frac{1}{E}])$ is an isomorphism. As such the isomorphism
$$\mathrm{Hom}_{\mathfrak{S}[\frac{1}{E}]}(M \otimes_\varphi \mathfrak{S}[\tfrac{1}{E}], N \otimes_\varphi \mathfrak{S}[\tfrac{1}{E}]) \to \mathrm{Hom}_{\mathfrak{S}[\frac{1}{E}]}(M[\tfrac{1}{E}], N[\tfrac{1}{E}])$$
which sends $f \mapsto \varphi_N \circ f \circ \varphi_M^{-1}$ makes $\mathrm{Hom}(M,N)$ into a Breuil–Kisin module.

With these definitions the evaluation map $\mathrm{Hom}(M,N) \otimes M \to N$ describes a morphism of Breuil–Kisin modules.

NOTATION 2.4.4. Recall that if $\mathcal{O}_{\mathcal{E}}$ is the p-adic completion of $\mathfrak{S}[\frac{1}{u}]$ then an etale φ-module over $\mathcal{O}_{\mathcal{E}}$ is a finitely generated $\mathcal{O}_{\mathcal{E}}$-module N^{et} equipped with an isomorphism $N^{\mathrm{et}} \otimes_{\varphi, \mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}} \cong N^{\mathrm{et}}$.

PROPOSITION 2.4.5 (Fontaine). *The functor $N^{\mathrm{et}} \mapsto T(N^{\mathrm{et}}) = (N^{\mathrm{et}} \otimes_{\mathcal{O}_{\mathcal{E}}} W(C^\flat))^{\varphi=1}$ is an exact $\otimes$-equivalence between the category of etale φ-modules over $\mathcal{O}_{\mathcal{E}}$ and the category of finitely generated $\mathbb{Z}_p$-modules equipped with a continuous $\mathbb{Z}_p$-linear action of G_{K_∞}. The representation $T(N^{\mathrm{et}})$ is determined up to isomorphism by the existence of a φ, G_{K_∞}-equivariant isomorphism*
$$N^{\mathrm{et}} \otimes_{\mathcal{O}_{\mathcal{E}}} W(C^\flat) \cong T(N^{\mathrm{et}}) \otimes_{\mathbb{Z}_p} W(C^\flat)$$

PROOF. There are two parts to this proposition. First note that our embedding $\mathfrak{S} \to A_{\mathrm{inf}}$ modulo p induces an inclusion $k((u)) \to C^\flat$. This inclusion identifies u with $(\pi, \pi^{1/p}, \ldots)$ which is fixed by G_{K_∞}, and so the action of G_K on $C^\flat$ induces a homomorphism
$$\mathrm{G}_{K_\infty} \to \mathrm{Aut}(C^\flat / k((u))^{\mathrm{perf}})$$
Since $C^\flat$ is the completed algebraic closure of $k((u))$ we may identify the right hand side with $G_{k((u))}$. Since the completion of K_∞ is a perfectoid field in the sense of [**24**], whose tilt is the completed perfection of $k((u))$, this map is an isomorphism (e.g. by [**24**, Theorem 3.7]).

For the second part of the proposition we invoke the results of [**8**, Section A]. Note that the result stated here differs from its usual formulation in that the ring $W(C^\flat)$ is usually replaced with a smaller ring $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$, the completion of the maximal unramified extension of $\mathcal{O}_{\mathcal{E}}$. In [**8**, Propostion 1.2.6] the result is proven with $\mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$ in place of $W(C^\flat)$, so to prove our proposition we just need to show that

$$(N^{\mathrm{et}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\widehat{\mathcal{E}}^{\mathrm{ur}}})^{\varphi=1} \subset T(N^{\mathrm{et}}) = (N^{\mathrm{et}} \otimes_{\mathcal{O}_{\mathcal{E}}} W(C^\flat))^{\varphi=1}$$

is an equality. For this it suffices to consider the case when N^{et} is killed by p. In this case the equality follows because for any algebraically closed field F of characteristic p and any finite dimensional F-vector V equipped with a φ-semilinear bijection, $V^{\varphi=1}$ is a finite dimensional $\mathbb{F}_p$-vector space of dimension equal to the F-dimension of V. □

PROPOSITION 2.4.6. *There is an exact $\otimes$-functor $M \mapsto T(M) = (M \otimes_{\mathfrak{S}} W(C^\flat))^{\varphi=1}$ from $\mathrm{Mod}_K^{\mathrm{BK}}$ to the category of finitely generated $\mathbb{Z}_p$-modules equipped with a continuous $\mathbb{Z}_p$-linear action of G_{K_∞}. The representation $T(M)$ is determined up to isomorphism by the existence of a φ, G_{K_∞}-equivariant identification*

$$M \otimes_{\mathfrak{S}} A_{\mathrm{inf}}[\tfrac{1}{\varphi^{-1}(\mu)}] \cong T(M) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}[\tfrac{1}{\varphi^{-1}(\mu)}]$$

PROOF. Since the map $\mathfrak{S} \to \mathcal{O}_{\mathcal{E}}$ is flat the functor $M \mapsto M^{\mathrm{et}} = M \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ is an exact $\otimes$-functor from Mod_K^{BK} to the category of etale φ-modules, and so from Proposition 2.4.5 we deduce that $M \mapsto T(M) = T(M^{\mathrm{et}})$ is an exact $\otimes$-functor as required.

To show that there exists a φ, G_{K_∞}-equivariant identification as stated we appeal to [**4**, Lemma 4.26]. A little care needs to be taken when doing this however; our embedding $\mathfrak{S} \to A_{\mathrm{inf}}$ is different from that of [**4**], which is obtained by composing our embedding with φ on A_{inf}. The upshot of this is that if M is a Breuil–Kisin module, $M \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ is not a Breuil–Kisin-Fargues module as defined in [**4**, §3.4]. However $M \otimes_{\varphi} A_{\mathrm{inf}}$ is a Breuil–Kisin-Fargues module. It is for this reason that $\varphi^{-1}(\mu)$ appears here where μ would appear in [**4**]. □

Our interest in Breuil–Kisin modules comes from the following theorem of Kisin. To state his result we must first introduce some notation.

NOTATION 2.4.7. Define $\mathcal{O}^{\mathrm{rig}}$ to be the subring of $K_0[[u]]$ consisting of power series which converge on the open unit disk. We equip $\mathcal{O}^{\mathrm{rig}}$ with the unique φ extending that on $\mathfrak{S} \subset \mathcal{O}^{\mathrm{rig}}$. The embedding of $\mathfrak{S}$ into A_{inf} extends to an embedding of $\mathcal{O}^{\mathrm{rig}}$ into B^+_{max}, and again this is compatible with all φ's. Inside $\mathcal{O}^{\mathrm{rig}}$ the following product converges.

$$\lambda = \prod_{n=0}^{\infty} \varphi^n(\tfrac{E(u)}{E(0)})$$

It can be shown that $\varphi(\lambda)$ is invertible in $A_{\max}$ (for instance see Lemma 9.1.3 below applied with $\nu = E([\pi^\flat]))$. Thus we can view $\mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}]$ as a subring of $B^+_{\max}$ compatibly with frobenius.

Let $\widehat{\mathfrak{S}}$ denote the E-adic completion of $\mathfrak{S}[\frac{1}{p}]$. Note that $\widehat{\mathfrak{S}}$ is a complete local ring with residue field $\mathfrak{S}[\frac{1}{p}]/E = K$; by Hensel's lemma it is therefore a K-algebra. The embedding $\mathfrak{S} \to \widehat{\mathfrak{S}}$ extends to an embedding $\mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}] \to \widehat{\mathfrak{S}}$ which sends a function onto its Taylor expansion around $u = \pi$. Since $E([\pi^\flat])$ is a uniformizer of B^+_{dR} the embedding of $\mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}]$ into $B^+_{\max}$ extends to an embedding of $\widehat{\mathfrak{S}} \to B^+_{\mathrm{dR}}$. This embedding is compatible with the filtrations, the filtration on $\widehat{\mathfrak{S}}$ being the E-adic filtration.

THEOREM 2.4.8 (Kisin). *There is a fully faithful covariant $\otimes$-functor $T \mapsto M(T)$ from the category of crystalline $\mathbb{Z}_p$-lattices into the full subcategory of Breuil–Kisin modules free over $\mathfrak{S}$. The module $M(T)$ admits functorial φ, G_{K_∞}-equivariant identifications:*

$$M(T) \otimes_{\mathfrak{S}} B_{\max} \cong T \otimes_{\mathbb{Z}_p} B_{\max} \cong D_{\mathrm{crys}}(V) \otimes_{K_0} B_{\max} \tag{2.4.9}$$

such that as submodules of (2.4.9) *we functorially have*

$$M(T)^\varphi \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{rig}}[\tfrac{1}{\varphi(\lambda)}] = D_{\mathrm{crys}}(V) \otimes_{K_0} \mathcal{O}^{\mathrm{rig}}[\tfrac{1}{\varphi(\lambda)}]$$

and

$$M(T) \otimes_{\mathfrak{S}} A_{\inf}[\tfrac{1}{\varphi^{-1}(\mu)}] = T \otimes_{\mathbb{Z}_p} A_{\inf}[\tfrac{1}{\varphi^{-1}(\mu)}] \tag{2.4.10}$$

This last identification characterises $M(T)$ up to isomorphism. Additionally as submodules of (2.4.9) *basechanged up to B_{dR} we have*

$$M(T) \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}} = F^0(D_{\mathrm{crys}}(V)_K \otimes_K \widehat{\mathfrak{S}}[\tfrac{1}{E}]) = \sum_i F^i D_{\mathrm{crys}}(V)_K \otimes E^{-i}\widehat{\mathfrak{S}}$$

SKETCH OF PROOF. The functor $T \mapsto M(T)$ was constructed in [**20**]. Since this formulation is a slight variant of what appears in [**20**] we sketch how the results of loc. cit. allow us to deduce the above.

First assume that $F^0 D_K = D_K$ where $D = \mathrm{D}_{\mathrm{crys}}(V)$ and $D_K = D \otimes_{K_0} K$. Associated to D is a finite free $\mathcal{O}^{\mathrm{rig}}$-module $\mathcal{M}(D)$ equipped with an injection $\mathcal{M}(D) \otimes_\varphi \mathcal{O}^{\mathrm{rig}} \to \mathcal{M}(D)$ which becomes an isomorphism after inverting λ, see [**20**, Lemma 1.2.2]. The module $\mathcal{M}(D)$ is constructed, using the filtration on D_K, as an $\mathcal{O}^{\mathrm{rig}}$-lattice inside $D \otimes_{K_0} \mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$ in such a way that

$$D \otimes_{K_0} \mathcal{O}^{\mathrm{rig}} \subset \mathcal{M}(D)^\varphi \tag{2.4.11}$$

with cokernel killed by a power of $\varphi(\lambda)$ (see [**20**, Lemma 1.2.6]), and such that as submodules of $\mathcal{M}(D) \otimes_{\mathcal{O}^{\mathrm{rig}}} B_{\mathrm{dR}} = D \otimes_{K_0} B_{\mathrm{dR}}$,

$$F^0(D_K \otimes_K \widehat{\mathfrak{S}}[\tfrac{1}{E}]) = \sum_i F^i D_K \otimes_K E^{-i}\widehat{\mathfrak{S}} = \mathcal{M}(D) \otimes_{\mathcal{O}^{\mathrm{rig}}} \widehat{\mathfrak{S}} \tag{2.4.12}$$

(see [**20**, Lemma 1.2.1]). Further Kisin shows that, as a consequence of the fact that D is admissible (i.e. is in the image of $V \mapsto D_{\mathrm{crys}}(V)$), there exists a Breuil–Kisin module M such that $\mathcal{M}(D) = M \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{rig}}$, see [**20**, Lemma 1.3.13].

A straightforward twisting argument shows that the above statements remain valid without the assumption that $F^0D = D$: if $F^0D \neq D$ then define $D(-n) = D$ with the shifted filtration $F^iD(-n)_K = F^{i-n}D_K$, and define $\mathcal{O}^{\mathrm{rig}}(n) = \mathcal{O}^{\mathrm{rig}}$ equipped with the semilinear map given by $\varphi_{\mathcal{O}^{\mathrm{rig}}(n)} = E^{-n}$. For n large enough that $F^0D(-n)_K = D(-n)_K$ one sets $\mathcal{M}(D) = \mathcal{M}(D(-n)) \otimes \mathcal{O}^{\mathrm{rig}}(n)$. Then still $\mathcal{M}(D)$ satisfies each of (2.4.11) and (2.4.12) and is such that $\mathcal{M}(D) = M \otimes \mathcal{O}^{\mathrm{rig}}$ for some Breuil–Kisin module M.

Let us remark that the M as above is not unique; in particular it is not necessarily the $M(T)$ of the theorem. Instead one finds $M(T)$ as an $\mathfrak{S}$-lattice inside $M[\frac{1}{p}]$ as we now explain. Note that for such an M all of the identifications of the theorem follow from the above except (2.4.10). Since λ is a unit in $B_{\max}$ we can base change the identification $M \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}] = D \otimes_{K_0} \mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$ to obtain φ-equivariant identifications

$$M \otimes_{\mathfrak{S}} B_{\max} \cong D \otimes_{K_0} B_{\max} \cong V \otimes_{\mathbb{Q}_p} B_{\max} \tag{2.4.13}$$

Each of these identifications are compatible with the G_{K_∞}-actions. As a consequence of (2.4.12) we see that

$$M \otimes_{\mathfrak{S}} B_{\mathrm{dR}}^+ = F^0(D \otimes_{K_0} B_{\mathrm{dR}}) = V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+$$

as submodules of (2.4.13) basechanged up to B_{dR}. As $M \otimes_{\mathfrak{S}} B_{\max} = V \otimes_{\mathbb{Q}_p} B_{\max}$ it follows that $M \otimes_{\mathfrak{S}} F^0B_{\max} = V \otimes_{\mathbb{Q}_p} F^0B_{\max}$, where $F^0B_{\max} = B_{\max} \cap B_{\mathrm{dR}}^+$. There is an exact sequence $0 \to \mathbb{Q}_p \to B_{\max}^{\varphi=1} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \to 0$ [**6**, Proposition III.3.1.] and so $(F^0B_{\max})^{\varphi=1} = \mathbb{Q}_p$. Therefore $V = (M \otimes_{\mathfrak{S}} F^0B_{\max})^{\varphi=1}$. Since $A_{\inf}[\frac{1}{\varphi^{-1}(\mu)}] \subset F^0B_{\max}$ we have that $T' = (M \otimes_{\mathfrak{S}} A_{\inf}[\frac{1}{\varphi^{-1}(\mu)}])^{\varphi=1} \subset V$. By Proposition 2.4.6 we have that

$$M \otimes_{\mathfrak{S}} A_{\inf}[\tfrac{1}{\varphi^{-1}(\mu)}] = T' \otimes_{\mathbb{Z}_p} A_{\inf}[\tfrac{1}{\varphi^{-1}(\mu)}]$$

as submodules of (2.4.13). As a consequence the $\mathbb{Z}_p$-rank of T' equals the $\mathfrak{S}$-rank of M, which equals the $\mathbb{Q}_p$-dimension of V; thus T' is a $\mathbb{Z}_p$-lattice inside V. If $T = T'$ then we take $M = M(T)$. If not the previous identification still allows us to identify φ-equivariantly $M \otimes_{\mathfrak{S}} \mathcal{E}$ with the etale φ-module $N^{\mathrm{et}}(V)$ over $\mathcal{E}$ associated to $V|_{G_{K_\infty}}$. Here $\mathcal{E} = \mathcal{O}_{\mathcal{E}}[\frac{1}{p}]$ where, as in 2.4.4, $\mathcal{O}_{\mathcal{E}}$ is the p-adic completion of $\mathfrak{S}[\frac{1}{u}]$. If $N^{\mathrm{et}}(T) \subset N^{\mathrm{et}}(V)$ is the etale φ-module over $\mathcal{O}_{\mathcal{E}}$ associated to $T|_{G_{K_\infty}}$ then we set $M(T) = N^{\mathrm{et}}(T) \cap M[\frac{1}{p}]$. One can show that $M(T)$ is finite free over $\mathfrak{S}$ and is therefore a Breuil–Kisin module which satisfies all the above properties.

The fact that the existence of an identification as in (2.4.10) characterises $M(T)$ up to isomorphism follows from a result of Kisin [**20**, Proposition 2.1.12] which implies that the functor $M \mapsto T(M)$ is fully faithful when restricted to Breuil–Kisin modules which are free over $\mathfrak{S}$. $\square$

Remark 2.4.14. Combining (2.4.10) with Proposition 2.4.6 shows that $T(M(T)) = T|_{G_{K_\infty}}$. Since the functor $M \mapsto T(M)$ is exact $T(M(T) \otimes \mathfrak{S}/p) = (T \otimes \mathbb{F}_p)|_{G_{K_\infty}}$. This means we can compute the reduction modulo p of T, at least after restricting to G_{K_∞}, by computing the reduction modulo p of $M(T)$.

Corollary 2.4.15. *Let T be a fixed crystalline $\mathbb{Z}_p$-lattice. Then $M \mapsto T(M)$ induces a bijection*

$$\left\{ \begin{array}{c} \textit{Breuil–Kisin modules } M \subset M(T)[\tfrac{1}{p}] \\ \textit{which are } \mathfrak{S}\textit{-finite free and have} \\ M(T)[\tfrac{1}{p}] = M[\tfrac{1}{p}] \end{array} \right\} \cong \left\{ \begin{array}{c} G_{K_\infty}\textit{-stable } \mathbb{Z}_p\textit{-lattices} \\ T \subset V \end{array} \right\}$$

Remark 2.4.16. It should be emphasised that $T \mapsto M(T)$ is *not* an exact functor. However we shall see below that it is exact after inverting p.

Definition 2.4.17. Let $\mathrm{Mod}_K^{\text{BK-iso}}$ denote the category of finite free $\mathfrak{S}[\frac{1}{p}]$-modules M equipped with an isomorphism

$$\varphi_M : M \otimes_{\varphi, \mathfrak{S}[\frac{1}{p}]} \mathfrak{S}[\tfrac{1}{pE}] \cong M[\tfrac{1}{E}]$$

such that φ-equivariantly $M = M^\circ[\frac{1}{p}]$ for some $M^\circ \in \mathrm{Mod}_K^{\text{BK}}$.

Remark 2.4.18. Note that any morphism in $\mathrm{Mod}_K^{\text{BK-iso}}$ is obtained from a morphism $M^\circ \to N^\circ$ in $\mathrm{Mod}_K^{\text{BK}}$ by inverting p. Since $M^\circ[\frac{1}{p}]$ is free over $\mathfrak{S}[\frac{1}{p}]$ for any $M^\circ \in \mathrm{Mod}_K^{\text{BK}}$ ([**4**, Proposition 4.3]) the category $\mathrm{Mod}_K^{\text{BK-iso}}$ can be identified with the isogeny category $\mathrm{Mod}_K^{\text{BK}} \otimes \mathbb{Q}_p$. Since $\mathrm{Mod}_K^{\text{BK}}$ is abelian it follows that $\mathrm{Mod}_K^{\text{BK-iso}}$ is abelian.

Proposition 2.4.19. *There is an exact fully faithful functor $V \mapsto M(V)$ from the category of crystalline representations into $\mathrm{Mod}_K^{\text{BK-iso}}$ such that $M^\circ \mapsto T(M^\circ)$ induces a bijection*

$$\left\{ \begin{array}{c} \textit{Breuil–Kisin modules } M^\circ \subset M(V) \\ \textit{which are } \mathfrak{S}\textit{-finite free and such} \\ \textit{that } M(V) = M^\circ[\tfrac{1}{p}] \end{array} \right\} \cong \left\{ \begin{array}{c} G_{K_\infty}\textit{-stable } \mathbb{Z}_p\textit{-lattices} \\ T \subset V \end{array} \right\}$$

Proof. The functor sends V onto $M(T)[\frac{1}{p}]$ for any crystalline $\mathbb{Z}_p$-lattice $T \subset V$. Using Corollary 2.4.15 we see that $M(T)[\frac{1}{p}]$ does not depend upon the choice of T. The only thing that needs to be checked is exactness of the functor $V \mapsto M(V)$. To see this we recall from Theorem 2.4.8 that there are functorial identifications

$$M(V) \otimes_{\mathfrak{S}[\frac{1}{p}]} \mathcal{O}^{\text{rig}}[\tfrac{1}{\lambda}] = D_{\text{crys}}(V) \otimes_{K_0} \mathcal{O}^{\text{rig}}[\tfrac{1}{\lambda}]$$

As $\mathfrak{S}[\frac{1}{p}]$ is a principal ideal domain the ring $\mathcal{O}^{\text{rig}}[\frac{1}{\lambda}]$ is flat over $\mathfrak{S}[\frac{1}{p}]$. In fact it is faithfully flat; any maximal ideal of $\mathfrak{S}[\frac{1}{p}]$ is generated by a polynomial with at least one zero on the open unit disk, on the other hand units in $\mathcal{O}^{\text{rig}}[\frac{1}{\lambda}]$ either have no zeroes, or infinitely many zeroes coming from λ's in the denominator. From this faithful flatness we deduce exactness of $V \mapsto M(V)$ from exactness of $V \mapsto D_{\mathrm{crys}}(V)$. $\square$

5. Crystalline Representations with Coefficients

Our interest in p-adic Galois representations does not extend to issues concerning rationality, and so it will be convenient for us to consider not all p-adic Galois representations but those which take values in $\mathrm{GL}_n(E)$ with E some sufficiently large finite extension of $\mathbb{Q}_p$. In this section we show how the above constructions should be adapted to accommodate this point of view.

Thus let us once and for all fix a coefficient field[3] $E/\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k_E. Additionally we write ϖ for any uniformizer of E. We shall always allow ourselves to assume that E is arbitrarily large. In particular we shall assume $K_0 \subset E$.

Definition 2.5.1. A continuous representation of G_K on an E-vector space V is crystalline if V, viewed as a $\mathbb{Q}_p$-vector space and so a p-adic Galois representation, is crystalline. If this is the case we say V is a crystalline E-representation. Functoriality of $V \mapsto D_{\mathrm{crys}}(V)$ then implies that the K_0-vector space $D_{\mathrm{crys}}(V)$ admits a φ-equivariant action of E and so in particular may be viewed as a module over $K_0 \otimes_{\mathbb{Q}_p} E$.

Remark 2.5.2. Recall that E is assumed to contain K_0. We typically reserve the symbol τ for an element of $\mathrm{Hom}_{\mathbb{Q}_p}(K_0, E) = \mathrm{Hom}_{\mathbb{F}_p}(k, k_E)$. The map $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O} \to \prod_{\tau : k \to k_E} \mathcal{O}$ given by $a \otimes b \mapsto (\tau(a)b)_\tau$ is an isomorphism. We let $i_\tau \in W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}$ denote the idempotent corresponding to the τ-th factor. Every $W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}$-module M may be decomposed as a product

$$M = \prod_{\tau : k \to k_E} M_\tau$$

where $M_\tau = i_\tau M$. The $\mathcal{O}$-module M_τ may be described as the subset of M on which the action of $W(k)$ coincides with the action of $\tau(W(k)) \subset \mathcal{O}$.

If V is a crystalline E-representation then applying this to $D_{\mathrm{crys}}(V)$ allows us to decompose this module as $\prod_\tau D_{\mathrm{crys}}(V)_\tau$.

Lemma 2.5.3. *If V is a crystalline E-representation then $D_{\mathrm{crys}}(V)$ is free as a module over $K_0 \otimes_{\mathbb{Q}_p} E$.*

Proof. Each $D_{\mathrm{crys}}(V)_\tau$ is a E-vector spaces and we have to show they all have the same E-dimension. This will follow if we can show that $\varphi : D_{\mathrm{crys}}(V) \to D_{\mathrm{crys}}(V)$ restricts to an E-linear map $\varphi : D_{\mathrm{crys}}(V)_{\tau \circ \varphi} \to D_{\mathrm{crys}}(V)_\tau$. Recall that $D_{\mathrm{crys}}(V)_\tau$ is the set of $v \in D_{\mathrm{crys}}(V)$ such that

[3] We apologize for our use of E for both the coefficient field and the minimal polynomial of π. In all cases it should be clear from the context which E we are referring to.

$(k \otimes 1)v = (1 \otimes \tau(k))v$ for all $k \in K_0$. Thus, if $v \in D_{\mathrm{crys}}(V)_{\tau\circ\varphi}$ and $k \in K_0$ then

$$(k \otimes 1)\varphi(v) = \varphi((\varphi^{-1}(k) \otimes 1)v) = \varphi((1 \otimes \tau(k))v) = (1 \otimes \tau(k))\varphi(v)$$

and so $\varphi(v) \in D_{\mathrm{crys}}(V)_\tau$. □

Remark 2.5.4. Lemma 2.5.3 tells us that $D_{\mathrm{crys}}(V)_K$ is free over $K \otimes_{\mathbb{Q}_p} E$. As the filtration on $D_{\mathrm{crys}}(V)_K$ is by $K \otimes_{\mathbb{Q}_p} E$-modules the decomposition $D_{\mathrm{crys}}(V)_K = \prod D_{\mathrm{crys}}(V)_{K,\tau}$ is a decomposition of filtered modules. Note however that the filtered pieces need not be free over $K \otimes_{\mathbb{Q}_p} E$.

Definition 2.5.5. For each $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, k_E)$ we define $\mathrm{HT}_\tau(V)$, the τ-th Hodge–Tate weights of V, to be the multiset of integers which contains i with multiplicity equal to the E-dimension of

$$\mathrm{gr}^i\, D_{\mathrm{crys}}(V)_{K,\tau}$$

Remark 2.5.6. If V is n-dimensional over E then $\mathrm{HT}(V)$ contains $n[E : \mathbb{Q}_p]$ integers, and is equal to the union

$$\bigcup_{\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k,k_E)} \bigcup_1^{[E:K_0]} \mathrm{HT}_\tau(V)$$

over all τ of $[E : K_0]$-copies of the multiset $\mathrm{HT}_\tau(V)$.

Arguing as in the proof of Proposition 2.3.3 we deduce:

Proposition 2.5.7. *Let V_i be two crystalline E-representations and let $\mathrm{Ext}^1(V_2, V_1)$ denote the first Yoneda extension group in the category of continuous representations of G_K on E-vector spaces. Then the subset $\mathrm{Ext}^1_{\mathrm{crys}}(V_2, V_1) \subset \mathrm{Ext}^1(V_2, V_1)$ consisting of classes represented by crystalline extensions is a subspace of E-dimension*

$$\sum_{\tau : k \to k_E} \mathrm{Card}\left(\{i - j < 0 \mid i \in \mathrm{HT}_\tau(V_1), j \in \mathrm{HT}_\tau(V_2)\}\right) + \dim_E \mathrm{Hom}_{E[G_K]}(V_2, V_1)$$

A crystalline $\mathcal{O}$-lattice T is a G_K-stable $\mathcal{O}_E$-lattice inside a crystalline E-representation. For such T the Breuil–Kisin module $M(T)$ admits a φ-equivariant action of $\mathcal{O}$ and is therefore a Breuil–Kisin module with $\mathcal{O}$-action as defined below:

Definition 2.5.8. A Breuil–Kisin module with $\mathcal{O}$-action is a pair (M, ι) where M is an object of $\mathrm{Mod}_K^{\mathrm{BK}}$ and ι is a nonzero $\mathbb{Z}_p$-algebra homomorphism $\iota : \mathcal{O} \to \mathrm{End}_{\mathrm{BK}}(M)$. Equivalently a Breuil–Kisin module M with $\mathcal{O}$-action is a finitely generated $\mathfrak{S}_{\mathcal{O}} = \mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}$-module equipped with an isomorphism

$$M \otimes_{\varphi, \mathfrak{S}_{\mathcal{O}}} \mathfrak{S}_{\mathcal{O}}[\tfrac{1}{E}] \cong M[\tfrac{1}{E}]$$

where φ here denotes the $\mathcal{O}$-linear extension of φ on $\mathfrak{S}$. We let $\mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ denote the category of such objects.

REMARK 2.5.9. As in Remark 2.4.3 the category $\mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ is an abelian category which admits natural notions of tensor product and internal hom. The tensor product of two objects has underlying $\mathfrak{S}_{\mathcal{O}}$-module $M \otimes_{\mathfrak{S}_{\mathcal{O}}} P$. The internal hom $\mathrm{Hom}(P, M)^{\mathcal{O}}$ has underlying its underlying module as $\mathrm{Hom}_{\mathfrak{S}_{\mathcal{O}}}(P, M)$ and is equipped with a frobenius just as in Remark 2.4.3.

REMARK 2.5.10. If M is an object of $\mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ then viewing M as a $\mathfrak{S}_{\mathcal{O}}$-module we can decompose

$$M = \prod_{\tau : k \to k_E} M_\tau$$

where each M_τ is a module over $(\mathfrak{S}_{\mathcal{O}})_\tau = \mathcal{O}[[u]]$. Arguing as in Lemma 2.5.3 shows that φ restricts to a map $\varphi : M_{\tau\circ\varphi} \to M_\tau[\frac{1}{E}]$ which is semilinear over $\mathcal{O}[[u]]$ for the endomorphism $\sum a_i u^i \mapsto \sum a_i u^{ip}$ (which we also write as φ).

As a consequence we deduce that if M is free as an $\mathfrak{S}$-module then M is free as a $\mathfrak{S}_{\mathcal{O}}$-module. Likewise, if M is killed by ϖ so that it is a module over $\mathfrak{S}_{\mathcal{O}}/\varpi = k[[u]] \otimes_{\mathbb{F}_p} k_E$ then, if M is free when viewed as a $k[[u]]$-module it is free as a $\mathfrak{S}_{\mathcal{O}}/\varpi$-module.

In the following proposition we write $A_{\mathcal{O}} = A \otimes_{\mathbb{Z}_p} \mathcal{O}$ for any $\mathbb{Z}_p$-algebra A. Analogously to Proposition 2.4.6 we have:

PROPOSITION 2.5.11. *There is an exact $\otimes$-functor $M \mapsto T_{\mathcal{O}}(M) = (M \otimes_{\mathfrak{S}_{\mathcal{O}}} W(C^\flat)_{\mathcal{O}})^{\varphi=1}$ from the category $\mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ to the category of finitely generated $\mathcal{O}$-modules equipped with a continuous $\mathcal{O}$-linear action of G_{K_∞}. The representation $T_{\mathcal{O}}(M)$ is determined up to isomorphism by the existence of a φ, G_{K_∞}-equivariant identification*

$$M \otimes_{\mathfrak{S}_{\mathcal{O}}} A_{\mathrm{inf}}[\tfrac{1}{\varphi^{-1}(\mu)}]_{\mathcal{O}} \cong T_{\mathcal{O}}(M) \otimes_{\mathcal{O}} A_{\mathrm{inf}}[\tfrac{1}{\varphi^{-1}(\mu)}]_{\mathcal{O}}$$

Moreover there are G_{K_∞}-equivariant identifications $T(M) \cong T_{\mathcal{O}}(M)$.

PROOF. The first part follows by an argument identical to that used to prove Proposition 2.4.6. For the comparison of $T(M)$ and $T_{\mathcal{O}}(M)$ use that the natural map $M \otimes_{\mathfrak{S}} W(C^\flat) \to M \otimes_{\mathfrak{S}_{\mathcal{O}}} W(C^\flat)_{\mathcal{O}}$, which is clearly φ, G_{K_∞}-equivariant, is an isomorphism. $\square$

CHAPTER 3

Torsion representations of G_K and G_{K_∞}

We maintain the notation of Chapter 2, so that K is a finite extension of $\mathbb{Q}_p$ with residue field k and K_∞ is the extension of K obtained by adjoining a compatible system of p-th power roots of a fixed uniformiser $\pi \in K$ (see the beginning of Section 2.4.2).

The purpose of this chapter is twofold. Firstly we recall the structure of irreducible p-torsion representations of G_K (and also G_{K_∞}). Secondly, motivated by the fact that Breuil–Kisin modules give rise to representations of G_{K_∞} we prove a result concerning restriction of representations from G_K to G_{K_∞}. We shall show that restriction is fully faithful if one restricts attention to certain cyclotomic-free torsion representations.

1. Preliminaries

If H is a closed subgroup of a profinite group G and V is a discrete H-module (i.e. an abelian group equipped with the discrete topology and a continuous $\mathbb{Z}$-linear action of H) then the (co-)induced module

$$\mathrm{Ind}_H^G V = \{\text{continuous } f : G \to V \text{ such that } f(hg) = h \cdot f(g) \text{ for all } h \in H, g \in G\}$$

is a discrete G-module via right translation.

If V and W are two G-modules let $\mathrm{Hom}(V, W)$ for the set of $\mathbb{Z}$-linear homomorphisms $V \to W$, equipped with the usual G-action $g(f)(v) = g(f(g^{-1}(v)))$. If $V = V^U$ for some open subgroup $U \subset G$ then $\mathrm{Hom}(V, W)$ is a (discrete) G-module. This is particular true if V is finite. We have

Lemma 3.1.1. *Let V be a G-module satisfying $V = V^U$ for some open subgroup $U \subset G$ and let W be any H-module W. Then there are G-equivariant isomorphisms*

$$\mathrm{Hom}(V, \mathrm{Ind}_H^G W) = \mathrm{Ind}_H^G \mathrm{Hom}(V|_H, W)$$

These are functorial in V and W.

Remark 3.1.2. By functoriality the lemma remains true if for any ring A we suppose V and W are A-modules with A-linear actions of G and H, and if we take $\mathrm{Hom}(-,-)$ to be the set of A-linear homomorphisms. In practice we will always apply the lemma in this setting.

We shall also make use of the projection formula which says that if V and W are A-modules with continuous (for the discrete topology) A-linear actions of G and H respectively then there exist functorial isomorphisms

$$V \otimes_A \mathrm{Ind}_H^G W = \mathrm{Ind}_H^G(V|_H \otimes_A W)$$

We shall slightly abuse notation by writing Ind_L^K in place of $\mathrm{Ind}_{\mathrm{Gal}(*/L)}^{\mathrm{Gal}(*/K)}$ if L/K is a finite extension; in all cases the field $*$ will be evident from the context.

2. Irreducible Torsion Representations

Recall our coefficient field E, with ring of integers $\mathcal{O}$ and residue field k_E. Unless otherwise mentioned, by a representation we mean a continuous representation on a finite dimensional k_E-vector space. Recall we are assuming E is arbitrarily large so that in particular the following lemma holds for any given representation.

LEMMA 3.2.1. *Let G be a finite group and suppose that $I \subset G$ is a normal abelian subgroup of order prime to p such that G/I is abelian and $1 \to I \to G \to G/I \to 1$ is split. Then any irreducible representation V of G on a k_E-vector space is induced from a character $H \to k_E^\times$ for some subgroup $H \subset G$ with $I \subset H$.*

PROOF. Our assumption that I is abelian of order prime to p and that k_E is large means that we can decompose $V|_I = \bigoplus_\chi V_\chi$ as a sum of subspaces on which I acts by characters $\chi : I \to k_E^\times$. The group G/I acts on the set of such χ: if $v \in V_\chi$ then $\gamma v \in V_{\chi^{(\gamma)}}$ where $\chi^{(\gamma)}(x) = \chi(\gamma^{-1} x \gamma)$. Let us fix a χ appearing in $V|_I$ and let $I \subset H \subset G$ be the subgroup corresponding to the stabiliser of χ in G/I. Note that $[G : H] \le \dim_{k_E} V$ by the orbit-stabiliser theorem.

Frobenius reciprocity (Lemma 3.1.1) implies there exists a nonzero map $V|_H \to \mathrm{Ind}_I^H \chi$. Since G is a semidirect product it is easy to see that χ being fixed by the action of H/I implies χ extends to a character $\chi : H \to k_E^\times$. This implies $\mathrm{Ind}_I^H \chi = \chi \otimes R$ where R denotes the regular representation of H/I. As H/I is abelian all irreducible representations of H/I are one-dimensional and so R admits a composition series $R_n \subset \ldots \subset R_0 = R$ such that each R_i/R_{i+1} is one-dimensional. If i is the largest integer such that $V|_H \to \mathrm{Ind}_I^H \chi$ factors through $\chi \otimes R_i$ then $V|_H \to \chi \otimes R_i/R_{i+1}$ is nonzero. Letting $\chi' = \chi \otimes R_i/R_{i+1}$ and applying Frobenius reciprocity there exists a nonzero map $V \to \mathrm{Ind}_H^G \chi'$ which, V being irreducible, must be injective. We conclude that $[G : H] = \dim_{k_E} \mathrm{Ind}_H^G \chi'$ is $\ge \dim_{k_E} V$. The inequality of the first paragraph implies $[G : H] = \dim_{k_E} V$ and so this map is an isomorphism. □

REMARK 3.2.2. Recall that if L/K is any finite Galois extension then we obtain a filtration by normal subgroups $P(L/K) \subset I(L/K) \subset \mathrm{Gal}(L/K)$ by asserting that $\sigma \in I(L/K)$ (respectively $\sigma \in P(L/K)$) if and only if for all $x \in \mathcal{O}_L$ and any choice of uniformiser $\pi_L \in L$

$$\sigma(x) - x \in \pi_L \mathcal{O}_L, \qquad \text{(respectively } \sigma(x) - x \in \pi_L^2 \mathcal{O}_L)$$

The map $\sigma \mapsto \sigma(\pi_L)/\pi_L$ defines an injection $I(L/K)/P(L/K) \to k_L^\times$ and the image is the subgroup $\mu_e(k_L^\times) = \{z \in K_L \mid z^e = 1\}$ where e is the order of $I(L/K)/P(L/K)$.

The extension L/K is unramified if $I(L/K) = 1$, and tamely ramified if $P(L/K) = 1$. Let K^{ur} denote the maximal unramified extension of K, i.e., the union of all finite unramified extensions of K, and let K^{t} denote the maximal tamely ramified extension of K.

LEMMA 3.2.3. *Any irreducible representation of* $\mathrm{Gal}(K^{\mathrm{t}}/K)$ *is induced from a character over an unramified extension of* K.

PROOF. We would like to put ourselves in the situation of Lemma 3.2.1. Firstly our representation factors through some finite quotient $G = \mathrm{Gal}(L/K)$ with L/K tamely ramified. Being tamely ramified $I = I(L/K)$ embeds into $k_L^\times$ and so is abelian (even cyclic) of order prime to p. As the quotient G/I may be identified with $\mathrm{Gal}(k_L/k)$ it is cyclic. However it is not always the case that G will be a semidirect product of I and G/I. To fix this observe that we may replace L by any tamely ramified extension of L. We claim if L is sufficiently large then G will be a semidirect product of I and G/I. If this is possible then we can apply Lemma 3.2.1 to deduce the result.

Replacing L by a totally tamely ramified extension of L we can arrange that $e = e(L/K)$ is divisible by $\mathrm{Card}(k_L^\times)$, so that $\mu_e(k_L^\times) = k_L^\times$. Since L/K is tamely ramified $P(L/K) = 1$ which means the map $\sigma \mapsto \sigma(\pi_L)/\pi_L$ is injective and so defines an isomorphism $I \to \mu_e(k_L) = k_L^\times$. With this assumption we prove that G is a semidirect product by showing that $H^2(G/I, I) = 0$, the action of G/I on I being given by conjugation.

Note that the isomorphism $I \to k_L^\times$ is compatible with the action of G/I with G/I acting on $k_L^\times$ via the identification $G/I = \mathrm{Gal}(k_L/k)$. Since G/I is cyclic $H^2(G/I, I)$ may be described explicitly as

$$H^2(G/I, I) = (k_L^\times)^{G/I}/N(k_L^\times) = k^\times/N(k_L^\times)$$

where $N : k_L^\times \to k^\times$ denotes the norm map. Norm maps between finite fields are surjective so $H^2(G/I, I) = 0$ as was claimed. $\square$

We obtain the following well-known result:

PROPOSITION 3.2.4. (1) *Any irreducible representation of G_K on a k_E-vector space is induced from a character over an unramified extension.*
(2) *Any irreducible representation of G_{K_∞} on a k_E-vector space is induced from a character over $L_\infty = K_\infty L$ with L an unramified extension of K.*

PROOF. The subgroup $\mathrm{Gal}(\overline{K}/K^{\mathrm{t}})$ of G_K is pro-p and so the subspace of V fixed by this subgroup is non-zero. As $\mathrm{Gal}(\overline{K}/K^{\mathrm{t}})$ is normal in G_K this subspace is a G_K-stable subspace and so, since V is irreducible, the action of $\mathrm{Gal}(\overline{K}/K^{\mathrm{t}})$ on V must be trivial. As such we can view V as an irreducible representation of $\mathrm{Gal}(K^{\mathrm{t}}/K)$ and apply Lemma 3.2.3. This proves (1).

For (2) set $K^{\mathrm{t}}_\infty = K^{\mathrm{t}}K_\infty$. Then $\mathrm{Gal}(\overline{K}/K^{\mathrm{t}}_\infty)$ is a subgroup of $\mathrm{Gal}(\overline{K}/K^{\mathrm{t}})$ and so is pro-p. Thus the action of G_{K_∞} on any irreducible representation V of G_{K_∞} factors though $\mathrm{Gal}(K^{\mathrm{t}}_\infty/K_\infty)$. Since K_∞ is totally wildly ramified we have $K^{\mathrm{t}} \cap K_\infty = K$. As such restriction induces an isomorphism $\mathrm{Gal}(K^{\mathrm{t}}_\infty/K_\infty) \to \mathrm{Gal}(K^{\mathrm{t}}/K)$. This isomorphism identifies $\mathrm{Gal}(K^{\mathrm{t}}_\infty/L_\infty)$ with $\mathrm{Gal}(K^{\mathrm{t}}/L)$ and so (2) follows from (1). $\square$

LEMMA 3.2.5. *Let L/K be an unramified extension. Let W be a one-dimensional representation of G_L on a k_E-vector space and let $V = \mathrm{Ind}_L^K W$. Then the restriction map $H^0(G_K, V) \to H^0(G_{K_\infty}, V)$ is an isomorphism.*

PROOF. If $L_\infty = K_\infty L$ then $G_L \cap G_{K_\infty} = G_{L_\infty}$ and so restriction of functions gives a map $V \to \mathrm{Ind}_{L_\infty}^{K_\infty} W$ of G_{K_∞}-representations which fits into a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & \mathrm{Ind}_{L_\infty}^{K_\infty} W \\ \downarrow & & \downarrow \\ W & \longrightarrow & W|_{G_{L_\infty}} \end{array}$$

whose vertical arrows are given by evaluation at 1. Since $K_\infty \cap L = K$ the inclusion $G_{K_\infty} \subset G_K$ induces an identification $G_{K_\infty}/G_{L_\infty} = G_K/G_L$. Hence $V \to \mathrm{Ind}_{L_\infty}^{K_\infty} W$ is an isomorphism of G_{K_∞}-representations. Passing to cohomology we obtain a commutative diagram

$$\begin{array}{ccc} H^i(G_K, V) & \longrightarrow & H^i(G_{K_\infty}, \mathrm{Ind}_{L_\infty}^{K_\infty} W) \\ \downarrow & & \downarrow \\ H^i(G_L, W) & \longrightarrow & H^i(G_{L_\infty}, W|_{G_{L_\infty}}) \end{array} \tag{3.2.6}$$

The vertical arrows are isomorphisms and the horizontal arrows are the restriction maps (see [**26**, Section 2.5]). Taking $i = 0$ we see that to prove the lemma it suffices to show that $H^0(G_L, W) \to H^0(G_{L_\infty}, W)$ is an isomorphism.

The G_L-action on W is given by a character $\chi : G_L \to k_E^\times$, we must show that if χ is trivial on G_{L_∞} then it is trivial on G_L. The kernel of χ corresponds to a tamely ramified extension of L. If χ is trivial on G_{L_∞} then this tame extension must be contained in L_∞ and so, as L_∞/L is totally wildly ramified, must be equal to L. Hence χ is trivial on G_L and we are done. $\square$

We record the following two corollaries:

COROLLARY 3.2.7. *If V is an irreducible G_K-representation then $V|_{G_{K_\infty}}$ is irreducible also.*

Proof. We know $V = \mathrm{Ind}_L^K W$ with W one dimensional. From the proof of Lemma 3.2.5 we know $V|_{G_{K_\infty}} = \mathrm{Ind}_{L_\infty}^{K_\infty} W$. As V is irreducible, for each $\gamma \in \mathrm{Gal}(L/K)$, the $W^{(\gamma)}$ are pairwise non-isomorphic. Thus they are pairwise non-isomorphic when restricted to G_{L_∞} (by Lemma 3.2.5 in the one dimensional case). This shows $\mathrm{Ind}_{L_\infty}^{K_\infty} W$ is irreducible. □

Corollary 3.2.8. *Let V and W be continuous irreducible representations of G_K. Then any G_{K_∞}-equivariant isomorphism $V \to W$ is G_K-equivariant.*

Proof. Assume there exists a G_{K_∞}-equivariant isomorphism so that $\dim_{k_E} W = \dim_{k_E} V$. Then there is an unramified extension L/K such that $V = \mathrm{Ind}_L^K V_1$ and $W = \mathrm{Ind}_L^K W_1$ with V_1 and W_1 each one dimensional. We have $W|_{G_L} = \bigoplus_\gamma W_1^{(\gamma)}$ with γ running over $\mathrm{Gal}(L/K)$. Using Lemma 3.1.1 (here we use $\mathrm{Hom}(-,-)$ to denote the set of k_E-linear homomorphisms) we have G_K-equivariant identifications

$$\mathrm{Hom}(W, V) = \mathrm{Ind}_L^K(\mathrm{Hom}(\bigoplus_\gamma W_1^{(\gamma)}, V_1)) = \bigoplus_\gamma \mathrm{Ind}_L^K \mathrm{Hom}(W_1^{(\gamma)}, V_1)$$

If $f \in \mathrm{Ind}_L^K \mathrm{Hom}(W_1^{(\gamma)}, V_1)$ is fixed by G_{K_∞} then Lemma 3.2.5 implies f is fixed by G_K which implies the corollary. □

3. Restriction

Let A be an Artin local ring with residue field k_E and let $\mathcal{C}$ denote the category of finitely generated A-modules equipped with a continuous (for the discrete topology) A-linear action of G_K. Throughout this section, if $V, W \in \mathcal{C}$, we shall write $\mathrm{Hom}(V, W)$ for the set of A-linear homomorphisms $V \to W$ with the usual G_K-action. Since V is finite we have that $\mathrm{Hom}(V, W) \in \mathcal{C}$.

Notation 3.3.1. In this section we consider the full subcategory $\mathcal{C}^{\mathrm{cyc}}$ of $\mathcal{C}$ whose objects are cyclotomic-free, i.e. those $V \in \mathcal{C}$ which admits a composition series $0 = V_n \subset \ldots \subset V_0 = V$ such that $V_i/V_{i+1} \otimes \mathbb{Z}_p(1)$ is not isomorphic to V_j/V_{j+1} for any $j > i$. In other words

$$H^0(G_K, \mathrm{Hom}(V_i/V_{i+1} \otimes \mathbb{Z}_p(1), V_j/V_{j+1})) = 0$$

for $j > i$. Pictorially, if V is a k_E-vector space we can express

$$V \sim \begin{pmatrix} \ddots & * & * \\ 0 & V_1/V_2 & * \\ 0 & 0 & V_0/V_1 \end{pmatrix}$$

and we ask that for each i no block above V_i/V_{i+1} is isomorphic to $V_i/V_{i+1} \otimes \mathbb{Z}_p(1)$. In particular cyclotomic-freeness is ruling out representations of the form $\left(\begin{smallmatrix} \chi_{\mathrm{cyc}} & * \\ 0 & 1 \end{smallmatrix}\right)$.

Note that in our definition of cyclotomic-freeness we do not require that every composition series of V satisfies the conditions describes in Notation 3.3.1. For instance if the cyclotomic character is not trivial modulo p then the representation $V = \left(\begin{smallmatrix} \chi_{\mathrm{cyc}} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is cyclotomic-free because the composition series $k_E \subset V$ is as in Notation 3.3.1. However the composition series $k_E(\chi_{\mathrm{cyc}}) \subset V$ is not as in Notation 3.3.1.

REMARK 3.3.2. $\mathcal{C}^{\mathrm{cyc}}$ is closed under subquotients. To see this suppose V is as in Notation 3.3.1 and $W \subset V$ is a G_K-stable submodule. Set $W_i = W \cap V_i$; then $W_i/W_{i+1} \hookrightarrow V_i/V_{i+1}$ so W_i/W_{i+1} is either zero or equal to V_i/V_{i+1}. It follows that, after re-indexing, the W_i form a filtration as in Notation 3.3.1. If we set $(V/W)_i = V_i/(W \cap V_i)$ then $(V/W)_i/(V/W)_{i+1} \hookrightarrow V_i/(V_{i+1} + (W \cap V_i))$, since $V_i/(V_{i+1} + (W \cap V_i))$ is a quotient of V_i/V_{i+1} it follows similarly that $(V/W)_i$ is a filtration of V/W as in Notation 3.3.1.

WARNING 3.3.3. While $\mathcal{C}^{\mathrm{cyc}}$ is closed under subquotients it is not closed under extensions; for instance any character is in $\mathcal{C}^{\mathrm{cyc}}$ but if $p = 2$ the cyclotomic character is trivial so the sum of two copies of a character is not in $\mathcal{C}^{\mathrm{cyc}}$.

Recall our assumption that k_E is large so that Proposition 3.2.4 holds for (all irreducible subquotients of) any object of $\mathcal{C}^{\mathrm{cyc}}$ we consider. The key input to the results of this section is the following. It has previously been observed with varying degrees of generality in e.g., [**7**], [**17**] and [**21**].

PROPOSITION 3.3.4. *Let V be an object of $\mathcal{C}$ and suppose that V admits a filtration $0 = V_n \subset \ldots \subset V_0 = V$ by G_K-stable submodules such that*

- *each $V_i/V_{i+1} = \mathrm{Ind}_{L_i}^{K} W_i$ with L_i/K unramified and W_i of rank one,*
- *and $H^0(G_K, V_i/V_{i+1} \otimes \mathbb{Z}_p(-1)) = 0$.*

Then the restriction map $H^i(G_K, V) \to H^i(G_{K_\infty}, V)$ is an isomorphism if $i = 0$ and is injective if $i = 1$.

Note we do not require the filtration in the proposition to be a composition series, so that the V_i/V_{i+1} need not be irreducible.

PROOF. We shall argue by induction on the length n of the filtration. If $n = 1$ then V is induced from a character. Thus the $i = 0$ part of the proposition follows from Lemma 3.2.5. For the $i = 1$ part we appeal to the diagram (3.2.6) with $i = 1$. This reduces the proposition to the case when V is one dimensional and not equal to the cyclotomic character. For such V the result is proven in [**17**, Lemma 5.4.2].

For general V consider the exact sequence $0 \to V_1 \to V \to V/V_1 \to 0$. Both V_1 and V/V_1 admit filtrations as in the proposition so we may assume inductively that the result holds for V_1 and V/V_1. Consider the following commutative diagram, whose rows are exact.

$$\begin{array}{ccccccccccc} 0 \to H^0(G_K,V_1) & \longrightarrow & H^0(G_K,V) & \longrightarrow & H^0(G_K,V/V_1) & \longrightarrow & H^1(G_K,V_1) & \longrightarrow & H^1(G_K,V) & \longrightarrow & H^1(G_K,V/V_1) \\ \downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow & & \downarrow & & \downarrow \\ 0 \to H^0(G_{K_\infty},V_1) & \to & H^0(G_{K_\infty},V) & \to & H^0(G_{K_\infty},V/V_1) & \to & H^1(G_{K_\infty},V_1) & \to & H^1(G_{K_\infty},V) & \to & H^1(G_{K_\infty},V/V_1) \end{array}$$

An easy diagram chase completes the argument. □

LEMMA 3.3.5. *Let V, W be objects of $\mathcal{C}$ admitting composition series $(V_i)_i$ and $(W_j)_j$ such that for all i, j*

$$H^0(G_K, \mathrm{Hom}(V_i/V_{i+1} \otimes \mathbb{Z}_p(1), W_j/W_{j+1})) = 0$$

Then $\mathrm{Hom}(V,W)$ *satisfies the conditions of Proposition 3.3.4.*

PROOF. First consider the filtration on $\mathrm{Hom}(V,W)$ given by $\mathrm{Hom}(V,W_i)$. Since the sequences

$$0 \to \mathrm{Hom}(V,W_{i+1}) \to \mathrm{Hom}(V,W_i) \to \mathrm{Hom}(V,W_i/W_{i+1})$$

are exact the subquotients of this filtration are subobjects of $\mathrm{Hom}(V,W_i/W_{i+1})$. This means that if we can prove the lemma when $W = W_i/W_{i+1}$ then we will have proved the lemma in general, so assume this is the case. Now $\mathrm{Hom}(V,W)$ also admits a filtration $\mathrm{Hom}(V/V_{n-i},W)$ and since the sequences

$$0 \to \mathrm{Hom}(V/V_{n-i},W) \to \mathrm{Hom}(V/V_{n-i+1},W) \to \mathrm{Hom}(V_{n-i}/V_{n-i+1},W)$$

are exact, the subquotients of this filtration are subobjects of $\mathrm{Hom}(V_{n-i}/V_{n-i+1},W)$. This reduces the lemma to the case with V and W both induced from characters. Write $V = \mathrm{Ind}_L^K \psi$ and $W = \mathrm{Ind}_F^K \rho$ with F and L unramified extensions of K and characters ψ and ρ. By Lemma 3.1.1 and Mackey's theorem we have that

$$\mathrm{Hom}(V,W) = \bigoplus_\gamma \mathrm{Ind}_{FL}^K \mathrm{Hom}(\psi^{(\gamma)}, \rho)$$

where γ runs over a finite subset of G_K. Thus $\mathrm{Hom}(V,W)$ admits a filtration whose subquotients are the summands on the right. If we can show that

$$H^0(G_K, \mathrm{Ind}_{FL}^K \mathrm{Hom}(\psi^{(\gamma)} \otimes \mathbb{Z}_p(1), \rho)) = 0 \tag{3.3.6}$$

then we will be done. If it is not zero then $\psi^{(\gamma)} \otimes \mathbb{Z}_p(1) = \rho$ on G_{FL}. Suppose for a contradiction that this equality holds. Then both $\psi^{(\gamma)}$ and ρ extend to characters of $G_{F\cap L}$. If $F \neq L$ then $G_{F\cap L}$ is a proper subgroup of one of G_F or G_L, and this contradicts the fact that both $V = \mathrm{Ind}_L^K \psi^{(\gamma)}$ and $W = \mathrm{Ind}_F^K \rho$ are irreducible. Thus $F = L$ and so $\mathrm{Ind}_L^K \psi^{(\gamma)} \otimes \mathbb{Z}_p(1) \cong \mathrm{Ind}_F^K \rho$. But then $V \otimes \mathbb{Z}_p(1) \cong W$ which is also a contradiction. We conclude that (3.3.6) holds which finishes the proof. □

LEMMA 3.3.7. *Suppose $V, W \in \mathcal{C}^{\mathrm{cyc}}$ and consider the restriction map $H^0(G_K, \mathrm{Hom}(V,W)) \to H^0(G_{K_\infty}, \mathrm{Hom}(V,W))$. If either V or W is irreducible then this map is an isomorphism.*

PROOF. Suppose that V is irreducible and let $(W_j)_j$ denote a G_K-composition series W as in Notation 3.3.1. Suppose that $f : V \to W$ is a non-zero G_{K_∞}-equivariant homomorphism. There will be a largest j such that f factors through $W_j \hookrightarrow W$; then the composite

$$g : V \xrightarrow{f} W_j \to W_j/W_{j+1}$$

will be non-zero. Every G_K-irreducible representation is irreducible as a G_{K_∞}-representation by Corollary 3.2.7. Thus g must be a G_{K_∞}-isomorphism, and so a G_K-isomorphism after Corollary 3.2.8. As such we obtain a G_{K_∞}-equivariant splitting $f \circ g^{-1}$ of

$$0 \to W_{j+1} \to W_j \to W_j/W_{j+1} \to 0 \tag{3.3.8}$$

Lemma 3.3.5 applied with $V = W_j/W_{j+1}$ and $W = W_{j+1}$ allows us to invoke Proposition 3.3.4 to deduce that there exists a G_K-splitting h of (3.3.8). We have $h - f \circ g^{-1} \in H^0(G_{K_\infty}, \mathrm{Hom}(W_j/W_{j+1}, W_{j+1}))$ and so using Proposition 3.3.4 again we deduce that $h - f \circ g^{-1}$ is G_K-equivariant. Since h is G_K-equivariant and g is a G_K-equivariant isomorphism it follows that f is G_K-equivariant.

Now we prove the result when W is irreducible. Let $f : V \to W$ be a non-zero G_{K_∞}-equivariant map. Since $\mathfrak{m}_A V \subset V$ is G_K-stable and in the kernel of f we can suppose f is a map of k_E-vector spaces. It is easy to see that if $V \in \mathcal{C}^{\mathrm{cyc}}$ is a k_E-vector space then $V^\vee = \mathrm{Hom}(V, k_E)$ is in $\mathcal{C}^{\mathrm{cyc}}$ also. The argument of the first paragraph therefore shows that $f^\vee : W^\vee \to V^\vee$ is G_K-equivariant, and so f is G_K-equivariant too.

∎

THEOREM 3.3.9. *Let $V, W \in \mathcal{C}^{\mathrm{cyc}}$. Then any G_{K_∞}-equivariant $f : V \to W$ isomorphism is G_K-equivariant.*

PROOF. We argue by induction on the length of V as a G_K-representation. If V is irreducible we appeal to Corollary 3.2.8. For general V let (V_i) be a composition series for V as in Notation 3.3.1. Lemma 3.3.7 implies $f : V_{n-1} \to W$ is G_K-equivariant and so $W_{n-1} := f(V_{n-1})$ is G_K-stable. We have the following commutative diagram whose rows are exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_{n-1} & \longrightarrow & V & \longrightarrow & V/V_{n-1} & \longrightarrow & 0 \\ & & \downarrow\wr & & \downarrow f & & \downarrow\wr & & \\ 0 & \longrightarrow & W_{n-1} & \longrightarrow & W & \longrightarrow & W/W_{n-1} & \longrightarrow & 0 \end{array}$$

Since W_{n-1} is a G_K-stable submodule of W, our inductive hypothesis allows us to assume the two outer vertical arrows are G_K-equivariant. Applying Lemma 3.3.5 with $V = V/V_{n-1}$ and $W = V_{n-1}$ shows that Proposition 3.3.4 applies, and so the restriction map $H^1(G_K, \mathrm{Hom}(V/V_{n-1}, V_{n-1})) \to H^1(G_{K_\infty}, \mathrm{Hom}(V/V_{n-1}, V_{n-1}))$ is injective. This implies there must exist a G_K-equivariant $h : V \to W$ fitting into the diagram. If not, identifying $V_{n-1} = W_{n-1}$ and $V/V_{n-1} = W/W_{n-1}$ as G_K-representations via the outer vertical maps of the diagram, we see that W and V represent distinct classes in $H^1(G_K, \mathrm{Hom}(V/V_{n-1}, V_{n-1}))$

which become equal in $H^1(G_{K_\infty}, \mathrm{Hom}(V/V_{n-1}, V_{n-1}))$. Clearly the map h must equal f which proves the theorem. $\square$

The previous theorem is all that will be required for future applications, but one could ask if it is possible to remove the requirement that f be an isomorphism. Our problem with proving this seems to arise in showing that kernels and images of G_{K_∞}-equivariant maps are G_K-stable. However if we assume that $\chi_{\mathrm{cyc}} \not\cong 1$ modulo p then we have the following argument.

THEOREM 3.3.10. *Let V and W be objects of $\mathcal{C}^{\mathrm{cyc}}$ and assume the cyclotomic character is non-trivial modulo p. Then the restriction map $H^0(G_K, \mathrm{Hom}(V, W)) \to H^0(G_{K_\infty}, \mathrm{Hom}(V, W))$ is an isomorphism.*[1]

PROOF. Let $(V_i)_i$ denote a G_K-composition series for V as in Notation 3.3.1. We argue by induction on the lengths of both V and W. Using Lemma 3.3.7 we may assume the statement is true whenever one of V or W is replaced by a representation with strictly smaller length. If every G_{K_∞}-equivariant element of $\mathrm{Hom}(V, W)$ factors through $V \to V/V_{n-1}$ then

$$H^0(G_{K_\infty}, \mathrm{Hom}(V, W)) = H^0(G_{K_\infty}, \mathrm{Hom}(V/V_{n-1}, W))$$

and the result would follow from our inductive hypothesis. Thus we may assume there is a non-zero G_{K_∞}-equivariant map $V_{n-1} \to W$. By Lemma 3.3.7 this map must be G_K-equivariant and we obtain an exact sequence

$$0 \to \mathrm{Hom}(V, V_{n-1}) \to \mathrm{Hom}(V, W) \to X \to 0$$

where X may be viewed as a G_K-stable submodule of $\mathrm{Hom}(V, W/V_{n-1})$. Passing to cohomology we obtain the following diagram (which commutes and has exact rows).

$$\begin{array}{ccccccccc}
0 \to & H^0(G_K, \mathrm{Hom}(V, V_{n-1})) & \longrightarrow & H^0(G_K, \mathrm{Hom}(V, W)) & \longrightarrow & H^0(G_K, X) & \longrightarrow & H^1(G_K, \mathrm{Hom}(V, V_{n-1})) \\
& \downarrow\wr & & \downarrow\alpha & & \downarrow\wr & & \downarrow \\
0 \to & H^0(G_{K_\infty}, \mathrm{Hom}(V, V_{n-1})) & \to & H^0(G_{K_\infty}, \mathrm{Hom}(V, W)) & \to & H^0(G_{K_\infty}, X) & \to & H^1(G_{K_\infty}, \mathrm{Hom}(V, V_{n-1}))
\end{array}$$

The inductive hypothesis implies the leftmost vertical arrow is an isomorphism. Since $W/V_{n-1} \in \mathcal{C}^{\mathrm{cyc}}$ we also have that $H^0(G_K, \mathrm{Hom}(V, W/V_{n-1})) \to H^0(G_{K_\infty}, \mathrm{Hom}(V, W/V_{n-1}))$ is an isomorphism and so, because $X \subset \mathrm{Hom}(V, W/V_{n-1})$, the same is true for the third vertical arrow above. If the rightmost vertical map is injective then that α is an isomorphism follows from a diagram chase. To establish this injectivity it suffices to check that Proposition 3.3.4 can be applied to $\mathrm{Hom}(V, V_{n-1})$. Note that $H^0(G_K, \mathrm{Hom}(V_i/V_{i+1} \otimes \mathbb{Z}_p(1), V_{n-1}) = 0$ for $i < n-1$ by hypothesis. Also, because we are assuming the cyclotomic character is non-zero, the same is true for $i = n-1$. Thus Lemma 3.3.5 implies Proposition 3.3.4 can be applied, which finishes the proof. $\square$

Let us conclude the section by giving an application of these results to G_K and G_{K_∞}-stable lattices inside extensions.

[1] If we took $k_E = \overline{\mathbb{F}}_p$ then this result could be stated in terms of full-faithfulness of restriction from G_K to G_{K_∞}.

NOTATION 3.3.11. If G is a topological group acting linearly and continuously on a topological abelian group M then we let $Z^1(G, M)$ denote the group of continuous 1-cocycles $G \to M$ and $B^1(G, M) \subset Z^1(G, M)$ the group of 1-coboundaries.

LEMMA 3.3.12. *Let W be a finite free $\mathcal{O}$-module equipped with a continuous $\mathcal{O}$-linear action of G_K. Let $\overline{W} = W \otimes_{\mathcal{O}} k_E$ and suppose that the conditions of Proposition 3.3.4 hold for $\overline{W}$.*

(1) *If $b \in B^1(G_K, W[\frac{1}{p}])$ is such that $b_\sigma \in W$ for all $\sigma \in G_{K_\infty}$ then $b \in Z^1(G_K, W)$.*
(2) *If $\overline{c} \in Z^1(G_K, \overline{W})$ is such that $\overline{c}_\sigma = 0$ for all $\sigma \in G_{K_\infty}$ then $\overline{c} = 0$.*

PROOF. (1) There is a $w \in W$ such that $\varpi^n b_\sigma = (\sigma - 1)w$ for some $n \geq 0$. If $n = 0$ then there is nothing to prove so assume otherwise. Since $b_\sigma \in W$ for $\sigma \in G_{K_\infty}$ we deduce that $\overline{w}$, the image of w in W/ϖ^n, is fixed by G_{K_∞}. Since the conditions of Proposition 3.3.4 holds for $\overline{W}$ they also hold for W/ϖ^n. Thus the $i = 0$ part of Proposition 3.3.4 applied to $\overline{W}/\varpi^n$ shows that $\overline{w}$ is then fixed by G_K and so $b_\sigma \in W$ for all $\sigma \in G_K$.

(2) If $\overline{c}$ is zero on G_{K_∞} then the $i = 1$ part of Proposition 3.3.4 implies the class $[\overline{c}] = 0$ in $H^1(G_K, \overline{W})$. Therefore $\overline{c}_\sigma = (\sigma - 1)\overline{w}$ is a 1-coboundary. We must have $\overline{w}$ fixed by G_{K_∞} and so the $i = 0$ part of Proposition 3.3.4 implies $\overline{w}$ is fixed by G_K. In other words $\overline{c} = 0$. $\square$

PROPOSITION 3.3.13. *Let $0 \to V_1 \to V \to V_2 \to 0$ be an exact sequence of continuous G_K-representations on finite dimensional E-vector spaces. Let $T \subset V$ be a G_{K_∞}-stable $\mathcal{O}$-lattice such that $T_1 = T \cap V_1$ and $T_2 = \mathrm{Im}(T) \subset V_2$ are both G_K-stable $\mathcal{O}$-lattices of V_1 and V_2 respectively. Let $W = \mathrm{Hom}(T_2, T_1)$ and $\overline{W} = W \otimes_{\mathcal{O}} k_E$. Suppose the conditions of Proposition 3.3.4 hold for $\overline{W}$. Then T is G_K-stable.*

PROOF. With T_1 and T_2 as defined T fits into an exact sequence $0 \to T_1 \to T \to T_2 \to 0$. As in Construction 2.3.1, choosing an $\mathcal{O}$-splitting of this sequence we obtain a continuous 1-cocycle $c : G_K \to W[\frac{1}{p}]$ in $Z^1(G_K, W[\frac{1}{p}])$ given by the composition

$$c_\sigma : T_2 \xrightarrow{\sigma^{-1}} T_2 \to T \xrightarrow{\sigma} V \to V_1$$

Since T is assumed G_{K_∞}-stable we have $c_\sigma \in W$ for $\sigma \in G_{K_\infty}$. Since both T_1 and T_2 are assumed G_K-stable we have that T is G_K-stable if and only if $c_\sigma \in W$ for all $\sigma \in G_K$.

Recall from [**28**, Proposition 2.3] that the natural map $H^1(G_K, W) \to H^1(G_K, W[\frac{1}{p}])$ (the H^1's here describing continuous group cohomology) induces an isomorphism $H^1(G_K, W)[\frac{1}{p}] = H^1(G_K, W[\frac{1}{p}])$. As such there exists $n \geq 0$ (which we shall assume to be minimal) such that, with c as in the previous paragraph, the class of $\varpi^n c$ in $H^1(G_K, W[\frac{1}{p}])$ equals a class represented by a cocycle in $Z^1(G_K, W)$. This means there is a $b \in B^1(G_K, W[\frac{1}{p}])$ such that $\varpi^n c_\sigma - b_\sigma \in W$ for all $\sigma \in G_K$. As $c_\sigma \in W$ for $\sigma \in G_{K_\infty}$ we have

$b_\sigma \in W$ for $\sigma \in G_{K_\infty}$. Thus Lemma 3.3.12 implies $b_\sigma \in W$ for all $\sigma \in G_K$ and so $\varpi^n c \in Z^1(G_K, W)$. We can therefore consider $\bar{c}$ the reduction of $\varpi^n c$ modulo ϖ. If $n \geq 1$ then $\bar{c}|_{G_{K_\infty}} = 0$ and so Lemma 3.3.12 tells us that $\bar{c} = 0$. This contradicts the minimality of n. Thus $n = 0$ which finishes the proof. $\square$

CHAPTER 4

Strongly Divisible Breuil–Kisin modules

In this chapter we introduce the notion of a strongly divisible Breuil–Kisin module and prove some basic results about their structure. A key result of Gee–Liu–Savitt says that the reduction modulo p of a Breuil–Kisin module associated to any crystalline representation with Hodge–Tate weights in $[0, p]$ will be strongly divisible.

REMARK 4.0.1. In later chapters we shall make the assumption that $K = K_0$. For this chapter this is not strictly necessary since all the objects we consider depend only on the residue field of K. However it is worth emphasising that when $K \neq K_0$ the categories we study in this section are not the correct ones from the point of view of the reduction of crystalline representations with Hodge–Tate weights in $[0, p]$ (at best they relate to crystalline representations with Hodge–Tate weights in $[0, p/e]$).

1. Filtrations

The language of filtered modules will be useful for us. Let A be a commutative ring equipped with a collection of ideals $(F^iA)_{i\in\mathbb{Z}}$ satisfying

$$F^{i+1}A \subset F^iA, \qquad (F^iA)(F^jA) \subset F^{i+j}A, \qquad F^iA = A \text{ for } i << 0$$

Let $\mathrm{Fil}(A)$ denote the category whose objects are A-modules equipped with a collection of A-submodules $(F^iM)_{i\in\mathbb{Z}}$ satisfying

$$F^{i+1}M \subset F^iM, \qquad (F^iA)(F^jM) \subset F^{i+j}M, \qquad F^iM = M \text{ for } i << 0$$

Morphisms in $\mathrm{Fil}(A)$ are A-module homomorphisms $f : M \to N$ which satisfy $f(F^iM) \subset F^iN$.

NOTATION 4.1.1. If M is an object of $\mathrm{Fil}(A)$ we set $\mathrm{gr}(M) = \bigoplus_i \mathrm{gr}^i(M)$ where $\mathrm{gr}^i(M) = F^iM/F^{i+1}M$. The module $\mathrm{gr}(A)$ admits an obvious structure of a ring and each $\mathrm{gr}(M)$ admits the structure of a module over $\mathrm{gr}(A)$.

REMARK 4.1.2. If M is an object of $\mathrm{Fil}(A)$ and $N \subset M$ is an A-submodule the induced filtration on N is that given by $F^iN = N \cap F^iM$. If $f : M \to N$ is a surjective A-module homomorphism the quotient filtration on N is that given by $F^iN = f(F^iM)$.

The category $\mathrm{Fil}(A)$ admits kernels and cokernels: if $f : M \to N$ is a morphism in $\mathrm{Fil}(A)$ then the A-submodule $\ker(f)$ with the induced filtration is the kernel in $\mathrm{Fil}(A)$. The A-module $\mathrm{coker}(f)$ with the quotient filtration arising from the map $N \to \mathrm{coker}(f)$ is the cokernel of f in $\mathrm{Fil}(A)$. It follows that $\mathrm{Fil}(A)$ admits images and coimages: $\mathrm{coim}(f) = \mathrm{coker}(\ker(f) \to M)$ and $\mathrm{Im}(f) = \ker(N \to \mathrm{coker}(f))$. Every morphism f factors as

$$M \to \mathrm{coim}(f) \to \mathrm{Im}(f) \to N$$

DEFINITION 4.1.3. A morphism $f : M \to N$ in $\mathrm{Fil}(A)$ is strict if $F^iN \cap f(M) = f(F^iM)$ for all $i \in \mathbb{Z}$. Equivalently f is strict if $\mathrm{coim}(f) \to \mathrm{Im}\, f$ is an isomorphism in $\mathrm{Fil}(A)$.

REMARK 4.1.4. The filtration on A induces the structure of a topological ring on A; the F^iA form a basis of open neighbourhoods of zero. Similarly the filtration on an object M of $\mathrm{Fil}(A)$ gives M the structure of a topological A-module. Then

- M is discrete if and only if $F^iM = 0$ for $i >> 0$;
- M is Hausdorff if and only if $\cap F^iM = 0$;
- M is complete if and only if the natural map $M \to \varprojlim M/F^iM$ is an isomorphism.

LEMMA 4.1.5. *Let $f : M \to N$ be a morphism in* $\mathrm{Fil}(A)$ *which is an isomorphism of A-modules.*

(1) *Then f is an isomorphism in* $\mathrm{Fil}(A)$ *if and only if* $\mathrm{gr}^i(M) \to \mathrm{gr}^i(N)$ *is injective for all i.*
(2) *If M is complete and N Hausdorff then f is an isomorphism in* $\mathrm{Fil}(A)$ *if and only if* $\mathrm{gr}^i(M) \to \mathrm{gr}^i(N)$ *is surjective for all i.*

PROOF. The following diagram commutes and has exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{i+1}M & \longrightarrow & F^iM & \longrightarrow & \mathrm{gr}^i(M) & \longrightarrow & 0 \\ & & \big\downarrow a & & \big\downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & F^{i+1}N & \longrightarrow & F^iN & \longrightarrow & \mathrm{gr}^i(N) & \longrightarrow & 0 \end{array}$$

Since $M \to N$ is an isomorphism of A-modules the leftmost and central vertical arrows are injective. The snake lemma gives an exact sequence $0 \to \ker c \to \mathrm{coker}(a) \to \mathrm{coker}(b) \to \mathrm{coker}(c)$.

If c is injective for all i one proves that each $F^iM \to F^iN$ is surjective by increasing induction on i; using as the base case the fact that $F^iM \to F^iN$ is surjective for $i << 0$, since $F^iM = M$ for $i << 0$.

For (2) we argue as in [**25**, Proposition 6]. Take $n_0 \in F^iN$. If c is surjective we can find $m_0 \in F^iM$ and $n_1 \in F^{i+1}N$ such that $b(m_0) = n_0 - n_1$. Repeating this construction with n_0 replaced with n_1 and so on, we obtain a sequence of elements $m_j \in F^{i+j}M, n_j \in F^{i+j}N$ such that $b(m_j) = n_j - n_{j+1}$. If M is complete then, as $F^iM \subset M$ is closed, $\sum m_j$ converges to an $m \in F^iM$. Since $\sum_{0 \le j \le n} b(m_j) = n_0 - n_{n+1}$ we have $b(m) - n_0 \in \cap_j F^jN$. As N is Hausdorff it follows that $b(m) = n_0$. □

LEMMA 4.1.6. *Let $f : M \to N$ be a morphism in* $\mathrm{Fil}(A)$. *Then the following are equivalent.*

(1) *f is strict;*

(2) $\mathrm{gr}(\ker(f)) \to \mathrm{gr}(M) \to \mathrm{gr}(N)$ *is exact;*

(3) $0 \to \mathrm{gr}(\ker(f)) \to \mathrm{gr}(M) \to \mathrm{gr}(N) \to \mathrm{gr}(\mathrm{coker}(f)) \to 0$ *is exact.*

If M is complete and N is Hausdorff then the same is true with (2) *replaced by*

($2'$) $\mathrm{gr}(M) \to \mathrm{gr}(N) \to \mathrm{gr}(\mathrm{coker}(f))$ *is exact for all i;*

PROOF. It is straightforward to check (without any conditions on M and N) that (2) is equivalent to $\mathrm{gr}^i \mathrm{coim}(f) \to \mathrm{gr}^i \mathrm{Im}(f)$ being injective for all i, that ($2'$) is equivalent to this map being surjective for all i, and that (3) is equivalent to this map being an isomorphism for all i. Thus the equivalence of (1),(2) and (3) follows from Lemma 4.1.5(1) applied to the morphism $\mathrm{coim}(f) \to \mathrm{Im}(f)$. Similarly using Lemma 4.1.5(2) one deduces the statement with (2) replaced with ($2'$), noting that M being complete implies $\mathrm{coim}(f)$ is complete and N being Hausdorff implies $\mathrm{Im}(f)$ is Hausdorff. □

LEMMA 4.1.7. *Let M be a Hausdorff object of* $\mathrm{Fil}(A)$ *with A complete. Suppose (m_j) is a finite collection of elements of M and suppose that there are integers r_j such that $m_j \in F^{r_j}M$. Let $\overline{m}_j$ denote the image of m_j in* $\mathrm{gr}^{r_j}(M)$. *If the $\overline{m}_j$ generate* $\mathrm{gr}(M)$ *over* $\mathrm{gr}(A)$ *then M is complete and the m_j generate M. Further*

$$F^iM = \sum_j (F^{i-r_j}A)m_j$$

If the $\overline{m}_j$ form a $\mathrm{gr}(A)$*-basis of* $\mathrm{gr}(M)$ *then the m_j are an A-basis of M.*

PROOF. Set $N = \bigoplus Am_j$ with $F^iN = \bigoplus (F^{i-r_j}A)m_j$. Then the surjection $N \to M$ is a morphism in $\mathrm{Fil}(A)$ with $\mathrm{gr}(N) \to \mathrm{gr}(M)$ surjective (respectively an isomorphism if the $\overline{m}_j$ form a $\mathrm{gr}(A)$-basis of $\mathrm{gr}(M)$). Since A is complete N is complete and so Lemma 4.1.6 implies $N \to M$ is a strict surjection (respectively an isomorphism) which prove the lemma. □

We now put ourselves in the following situation. Let $a \in A$ be a nonzerodivisor and equip A with the a-adic filtration (so $F^iA = a^iA$). Let M be a finite free A-module and let $N \subset M[\frac{1}{a}]$ be a finitely generated A-submodule with $N[\frac{1}{a}] = M[\frac{1}{a}]$. We make N into an object of $\mathrm{Fil}(A)$ by setting $F^iN = a^iM \cap N$.

LEMMA 4.1.8. *Suppose that A is complete. Give N/a the quotient filtration and suppose that a finite collection (g_i) of elements of N is given along with integers (r_i) such that $g_i \in F^{r_i}N$. If the images of g_i in* $\mathrm{gr}^{r_i}(N/a)$ *form a* $\mathrm{gr}(A/a) = A/a$*-basis of* $\mathrm{gr}(N/a)$ *then the (g_i) form a basis of N and the $(a^{-r_i}g_i)$ form a basis of M.*

PROOF. The induced filtration on the kernel aN of $N \to N/a$ is given by $F^i(aN) = aN \cap F^iN = aF^{i-1}N$ (because a is not a zerodivisor). Lemma 4.1.6 implies there is an exact sequence

$$0 \to \mathrm{gr}^{i-1}(N) \xrightarrow{a} \mathrm{gr}^i(N) \to \mathrm{gr}^i(N/a) \to 0 \qquad (4.1.9)$$

Thus $\mathrm{gr}(N)/a = \mathrm{gr}(N/a)$ where $a \in \mathrm{gr}(A)$ denotes the homogeneous element of degree 1 represented by $a \in A$. It is then easy to see (e.g. using the graded version of Nakayama's lemma) that the images of the g_i in $\mathrm{gr}(N)$ generate this module over $\mathrm{gr}(A)$. Since $\cap_i a^i \,\mathrm{gr}(A) = 0$ they are also $\mathrm{gr}(A)$-linearly independent. As N is finitely generated N is Hausdorff and so we may apply Lemma 4.1.7 to deduce that the (g_i) form an A-basis of N and that

$$F^n N = \sum (F^{n-r_i} A) g_i$$

As the g_i are A-linearly independent the $(a^{-r_i} g_i)$ are A-linearly independent. To show they generate M take $m \in M$ and n large enough that $a^n m \in N$. Then $a^n m \in F^n N$ and so $a^n m = \sum a_i g_i$ with $a_i \in F^{n-r_i} A$. It follows that $m = \sum (a^{r_i - n} a_i)(a^{-r_i} g_i)$ and so, since $(a^{r_i-n}) F^{n-r_i} A \subset A$, we are done. $\square$

Remark 4.1.10. As a consequence of Lemma 4.1.8 we see that there exists an A-basis (m_i) of M such that $(u^{r_i} m_i)$ is an A-basis of N if and only if for each i, $\mathrm{gr}^i(N/a)$ is finite free over A/a (the lemma only implies the if direction of this statement but the converse is easy to check).

Finally we give criteria to determine when two filtrations on a single module are the same (we apply this result only when A is a field).

Lemma 4.1.11. *Let M be an A-module equipped with two discrete filtrations $G^i M \subset F^i M$. Equip each $\mathrm{gr}^i_F(M)$ with the filtration induced by G and suppose that each $\mathrm{gr}^i_G(\mathrm{gr}^j_F(M))$ is A-projective of finite constant rank. Then each graded piece of F and G is A-projective of finite constant rank and*

$$\sum i \,\mathrm{rank}_A \,\mathrm{gr}^i_G(M) \le \sum i \,\mathrm{rank}_A \,\mathrm{gr}^i_F(M)$$

with equality if and only if $G = F$.

Proof. Observe that

$$H^{ij} := \mathrm{gr}^i_G(\mathrm{gr}^j_F(M)) = (F^j M \cap G^i M)/(F^{j+1} M \cap G^i M + F^j M \cap G^{i+1} M)$$

so by symmetry $H^{ij} = \mathrm{gr}^j_F(\mathrm{gr}^i_G(M))$. Choosing A-splittings of the exact sequences $0 \to G^{i+1}\,\mathrm{gr}^j_F(M) \to G^i\,\mathrm{gr}^j_F(M) \to H^{ij} \to 0$ allows us to (non-canonically) identify $\mathrm{gr}^j_F(M) = \bigoplus_i H^{ij}$ as A-modules. Similarly $\mathrm{gr}^i_G(M) = \bigoplus_j H^{ij}$. If follows that each graded piece of G and F are projective of finite constant rank and that

$$\sum_i i \,\mathrm{rank}_A \,\mathrm{gr}^i_G(M) = \sum_{i,j} i \,\mathrm{rank}_A H^{ij}, \qquad \sum_j j \,\mathrm{rank}_A \,\mathrm{gr}^j_F(M) = \sum_{i,j} j \,\mathrm{rank}_A H^{ij}$$

If $j \le i$ then $G^i M \subset F^j M$ and so $F^j \,\mathrm{gr}^i_G(M) = \mathrm{gr}^i_G(M)$. Thus $H^{ij} = 0$ for $j < i$ which implies $\sum i \,\mathrm{rank}_A \,\mathrm{gr}^i_G(M) \le \sum j \,\mathrm{rank}_A \,\mathrm{gr}^j_F(M)$. If we have equality then we must have $H^{ij} = 0$ for $j > i$. We finish the proof by showing this implies $G^j M = F^j M$ for each j. Let us induct on j; the base case is taken care of because, each of F and G being discrete, both $F^j M, G^j M$ are zero for large j. We are assuming $H^{ij} = 0$ for $j > i$ and so $H^{j-1,j} =$

$H^{j-2,j} = \ldots = 0$. Thus $G^j \operatorname{gr}^j_F(M) = G^{j-1} \operatorname{gr}^j_F(M) = \ldots = \operatorname{gr}^j_F(M)$ and so $G^j M + F^{j+1} M = F^j M$. By the inductive hypothesis $F^{j+1} M = G^{j+1} M$ and we conclude $G^j M = F^j M$. □

NOTATION 4.1.12. Say that a sequence of morphisms $M \to N \to P$ in $\mathrm{Fil}(A)$ is exact if it is exact as a sequence of A-modules and if $M \to N$ is strict. Lemma 4.1.6 implies that a sequence $0 \to M \to N \to P \to 0$ in $\mathrm{Fil}(A)$ which is exact in the category of A-modules is exact in $\mathrm{Fil}(A)$ if and only if $0 \to \operatorname{gr}(M) \to \operatorname{gr}(N) \to \operatorname{gr}(P) \to 0$ is an exact sequence of A-modules.

COROLLARY 4.1.13. *Let k be a field and let $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ be a sequence of finite dimensional discrete objects in $\mathrm{Fil}(k)$ which is exact in the category of k-vector spaces. If f (respectively g) is strict then*

$$\sum i \dim_k \operatorname{gr}^i(N) \leq \sum i \dim_k \operatorname{gr}^i(M) + \sum i \dim_k \operatorname{gr}^i(P) \quad \textit{(respectively } \geq\textit{)}$$

Conversely if one of f or g is strict then equality implies the sequence is exact in $\mathrm{Fil}(k)$.

PROOF. As P is discrete we can apply Lemma 4.1.11 to deduce that

$$\sum i \dim_k \operatorname{gr}^i(N/M) \leq \sum i \dim_k \operatorname{gr}^i(P)$$

with equality if and only if g is strict. If f is strict Lemma 4.1.6 tells us that $0 \to \operatorname{gr}(M) \to \operatorname{gr}(N) \to \operatorname{gr}(N/M) \to 0$ is exact, and so

$$\sum i \dim_k \operatorname{gr}^i(N) = \sum i \dim_k \operatorname{gr}^i(M) + \sum i \dim_k \operatorname{gr}^i(N/M)$$

The lemma follows when we assume f is strict. If g is strict we argue similarly. Lemma 4.1.11 implies

$$\sum i \dim_k \operatorname{gr}^i(\ker g) \geq \sum i \dim_k \operatorname{gr}^i(M)$$

with equality if and only if f is strict. As g is strict $0 \to \operatorname{gr}(\ker g) \to \operatorname{gr}(N) \to \operatorname{gr}(P) \to 0$ is exact and so

$$\sum i \dim_k \operatorname{gr}^i(N) = \sum i \dim_k \operatorname{gr}^i(\ker g) + \sum i \dim_k \operatorname{gr}^i(P)$$

and the result follows. □

2. Torsion Objects

NOTATION 4.2.1. We denote by $\mathrm{Mod}_k^{\mathrm{BK}}$ the full subcategory of the category of $\mathrm{Mod}_K^{\mathrm{BK}}$ (Definition 2.4.2) whose objects are modules which are free over $\mathfrak{S}/p = k[[u]]$.

REMARK 4.2.2. Observe that $E(u)$ is congruent to u^e modulo p (recall $e = [K : K_0]$). Thus an object of $\mathrm{Mod}_k^{\mathrm{BK}}$ is a finite free $k[[u]]$-module equipped with a φ-semilinear injection

$$\varphi : M \hookrightarrow M[\tfrac{1}{u}]$$

which becomes a bijection after inverting u. In particular, since $\mathcal{O}_{\mathcal{E}}/p = k((u))$ (Notation 2.4.4) an object of $\mathrm{Mod}_k^{\mathrm{BK}}$ is the same thing as a $k[[u]]$-lattice inside a p-torsion etale φ-module.

LEMMA 4.2.3. *The exact $\otimes$-functor $M \mapsto T(M)$ restricts to an essentially surjective functor from $\mathrm{Mod}_k^{\mathrm{BK}}$ to the category of continuous representations of G_{K_∞} on finite dimensional $\mathbb{F}_p$-vector spaces. Moreover, if $M \in \mathrm{Mod}_k^{\mathrm{BK}}$ and*

$$0 \to T_1 \to T(M) \to T_2 \to 0$$

is an exact sequence of G_{K_∞}-representations then there exists a unique exact sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

in $\mathrm{Mod}_k^{\mathrm{BK}}$ such that $T(M_i) = T_i$.

PROOF. The functor $N^{\mathrm{et}} \mapsto T(N^{\mathrm{et}})$ from p-torsion etale φ-modules to $\mathbb{F}_p$-representations of G_{K_∞} is an exact equivalence by Proposition 2.4.5. Thus, if T is a p-torsion G_{K_∞}-representation there is a p-torsion etale φ-module N^{et} such that $T(N^{\mathrm{et}}) = T$. Any $k[[u]]$-lattice M in N^{et} is an object of $\mathrm{Mod}_k^{\mathrm{BK}}$ (Remark 4.2.2) and $T(M) = T(N^{\mathrm{et}}) = T$.

For the second part, we can find an exact sequence of etale φ-modules $0 \to N_1^{\mathrm{et}} \to N^{\mathrm{et}} \to N_2^{\mathrm{et}} \to 0$ such that $T(N_i^{\mathrm{et}}) = T_i$ and such that M is a $k[[u]]$-lattice inside N^{et}. Taking $M_1 = M \cap N_1^{\mathrm{et}}$ and $M_2 = \mathrm{Im}(M) \cap N_2^{\mathrm{et}}$ shows an exact sequence as desired exists in $\mathrm{Mod}_k^{\mathrm{BK}}$. If $M_1 \subset M$ was another object of $\mathrm{Mod}_k^{\mathrm{BK}}$ giving rise to T_1 then M_1 must be a $k[[u]]$-lattice inside N_1^{et}. However since M/M_1 must be torsion-free as a $k[[u]]$-module there can only be one such M_1. $\square$

Lemma 4.2.3 is true only for p-torsion Breuil–Kisin modules. For instance for etale φ-modules N which are free over $\mathcal{O}_{\mathcal{E}}$, the assertion that there exists a Breuil–Kisin module M such that $M \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}} = N$ is a considerably restrictive assertion.

CONSTRUCTION 4.2.4. Let $M \in \mathrm{Mod}_k^{\mathrm{BK}}$. By a composition series for M we mean a filtration

$$0 = M_n \subset \ldots \subset M_0 = M$$

by sub-Breuil–Kisin modules such that each M_i/M_{i+1} is an irreducible object (that is admits no non-zero proper subobjects of $N \in \mathrm{Mod}_k^{\mathrm{BK}}$ such that the cokernel of $N \hookrightarrow M_i/M_{i+1}$ is $k[[u]]$-torsion free) of $\mathrm{Mod}_k^{\mathrm{BK}}$. After Lemma 4.2.3 being irreducible is equivalent to asking that $T(M_i/M_{i+1})$ is an irreducible G_{K_∞}-representation. Lemma 4.2.3 also implies that composition series for M are in bijection with composition series for $T(M)$.

WARNING 4.2.5. It is not the case that the set of irreducible factors of a composition series is independent of the choice of composition series. See the following example.

EXAMPLE 4.2.6. Let k be the quadratic extension of $\mathbb{F}_p$ and let $M = k[[u]] \otimes_{\mathbb{F}_p} k_E^2$. Writing $\mathrm{Hom}_{\mathbb{F}_p}(k, k_E) = \{\tau \circ \varphi, \tau\}$ the modules M_τ and $M_{\tau\circ\varphi}$ are free rank two modules over $k_E[[u]]$ with bases given by (e_τ, f_τ) and $(e_{\tau\circ\varphi}, f_{\tau\circ\varphi})$ respectively. We make M into an object of $\mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ (in fact an object of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$, see Notation 4.4.1) by setting

$$\varphi_M(e_{\tau\circ\varphi}, f_{\tau\circ\varphi}) = (e_\tau, f_\tau)\begin{pmatrix} u^p & 0 \\ 0 & 1 \end{pmatrix} \qquad \varphi_M(e_\tau, f_\tau) = (e_{\tau\circ\varphi}, f_{\tau\circ\varphi})\begin{pmatrix} \alpha & 1 \\ 0 & u \end{pmatrix}$$

where $\alpha \in k_E^\times$ is not equal to 1. On the other hand if $\alpha - x = 1$ then

$$\varphi_M(e_{\tau\circ\varphi} - xuf_{\tau\circ\varphi}, f_{\tau\circ\varphi}) = (e_\tau - xf_\tau, f_\tau)\begin{pmatrix} u^p & 0 \\ 0 & 1 \end{pmatrix}$$

$$\varphi_M(e_\tau - xf_\tau, f_\tau) = (e_{\tau\circ\varphi} - xuf_{\tau\circ\varphi}, f_{\tau\circ\varphi})\begin{pmatrix} 1 & 1 \\ 0 & \alpha u \end{pmatrix}$$

This shows that M admits two composition series and the irreducible factors of these two composition series are not permutations of each other.

3. Strong Divisibility

The aim of this section it to describe a full subcategory $\mathrm{Mod}_k^{\mathrm{SD}} \subset \mathrm{Mod}_k^{\mathrm{BK}}$ (Definition 4.3.7). The motivation comes from a result of Gee–Liu–Savitt (Theorem 4.5.1) which implies that if T is a crystalline $\mathbb{Z}_p$-lattice with Hodge–Tate weights in $[0, p]$ then $M(T)/p$ is an object of $\mathrm{Mod}_k^{\mathrm{SD}}$.

We call objects of $\mathrm{Mod}_k^{\mathrm{SD}}$ strongly divisible Breuil–Kisin modules. We shall justify this name by relating strongly divisible Breuil–Kisin modules to Fontaine–Laffaille theory (Chapter 6).

Once we have defined $\mathrm{Mod}_k^{\mathrm{SD}}$ the main result of this section is to show that this category is stable under subquotients (Proposition 4.3.13).

CONSTRUCTION 4.3.1. Let M be an object of $\mathrm{Mod}_k^{\mathrm{BK}}$. Recall that M^φ denotes the image $\varphi(M \otimes_{\varphi,\mathfrak{S}} \mathfrak{S})$ inside $M[\frac{1}{u}]$. In other words it is the $k[[u]]$-submodule of $M[\frac{1}{u}]$ generated by $\varphi(M)$. We equip M^φ with a filtration given by $F^iM^\varphi = M^\varphi \cap u^iM$. The association $M \mapsto M^\varphi$ describes a functor $\mathrm{Mod}_k^{\mathrm{BK}} \to \mathrm{Fil}(k[[u]])$.

In a similar way we equip M with a filtration by setting $F^iM = \{m \in M \mid \varphi(m) \in u^iM\}$. Again this construction is functorial in M. Observe that the semilinear injection

$$\varphi : M \hookrightarrow M^\varphi$$

is compatible with these two filtrations.

LEMMA 4.3.2. *Make $M_k = M/u$ and $M_k^\varphi = M^\varphi/u$ into objects of* $\mathrm{Fil}(k)$ *by equipping each with the quotient filtration coming from M and M^φ respectively. Then the injection $\varphi : M \hookrightarrow M^\varphi$ induces a semilinear map of filtered modules*

$$M_k \to M_k^\varphi$$

which is functorial in M and a k-semilinear bijection of vector spaces.

Proof. All that needs to be checked is that $\varphi : M \to M^\varphi$ becomes an isomorphism after reducing modulo u and as M_k and M_k^φ have the same dimension over k we only need to check surjectivity. By definition M^φ is the $k[[u]]$-module generated by $\varphi(M) \subset M[\frac{1}{u}]$ so surjectivity modulo u follows since φ is an automorphism on $k = k[[u]]/u$. $\square$

Definition 4.3.3. If M is an object of $\mathrm{Mod}_k^{\mathrm{BK}}$ we let $\mathrm{Weight}(M)$ be the multiset of integers which contains i with multiplicity equal to

$$\dim_k \mathrm{gr}^i(M_k^\varphi)$$

Lemma 4.3.4. *Let M be an object of $\mathrm{Mod}_k^{\mathrm{BK}}$. The following are equivalent:*

(1) *The map $M_k \to M_k^\varphi$ is an isomorphism of filtered modules.*
(2) *There exists a $k[[u]]$-basis (f_i) of M and integers (r_i) such that $(u^{r_i} f_i)$ is a $k[[u^p]]$-basis of $\varphi(M)$.*

Proof. Suppose first that $M_k \to M_k^\varphi$ is an isomorphism of filtered modules. We can find integers r_i and elements $g_i \in F^{r_i}M$ whose images in $\mathrm{gr}(M_k)$ form a k-basis. As the induced map $\mathrm{gr}(M_k) \to \mathrm{gr}(M_k^\varphi)$ is an isomorphism it follows that the images of $\varphi(g_i) \in \varphi(M)$ in $\mathrm{gr}(M_k^\varphi)$ form a k-basis. Applying Lemma 4.1.8 with $M = M$, $N = M^\varphi$ and $a \in A$ equal to $u \in k[[u]]$ proves that (1) implies (2) with $f_i = u^{-r_i}\varphi(g_i)$.

To prove (2) implies (1) we use the f_i to give explicit descriptions of the filtration on M_k^φ. Since $\varphi(M)$ generates M^φ over $k[[u]]$, if $m \in M^\varphi$ then there are $\alpha_i \in k[[u]]$ such that $m = \sum \alpha_i(u^{r_i} f_i)$. If $m \in F^j M^\varphi$ then, since the f_i form a basis of M, we must have $\alpha_i \in F^{j-r_i}k[[u]]$. Hence

$$F^j M^\varphi = \sum (F^{j-r_i}k[[u]])(u^{r_i} f_i)$$

and so $F^j M_k^\varphi = \sum_{r_i \geq j} k\overline{f}_i$ where $\overline{f}_i$ denotes the image of $u^{r_i} f_i$ in M_k^φ. If $g_i \in M$ is such that $\varphi(g_i) = u^{r_i} f_i$ then $g_i \in F^j M$ if $r_i \geq j$. If $\overline{g}_i$ denotes the image of g_i in M_k then since the map $M_k \to M_k^\varphi$ sends $\overline{g}_i \mapsto \overline{f}_i$, it induces surjections $F^j M_k \to F^j M_k^\varphi$. Thus $M_k \to M_k^\varphi$ is an isomorphism in $\mathrm{Fil}(k)$. $\square$

Remark 4.3.5. Note that if we have a basis as in (2) of Lemma 4.3.4 then the proof of the previous lemma shows that $\mathrm{gr}^j(M_k^\varphi) = \sum_{r_i = j} k\overline{f}_i$. Thus the multiset $\{r_i\}$ is equal to $\mathrm{Weight}(M)$.

Remark 4.3.6. Isomorphism classes of objects in $\mathrm{Mod}_k^{\mathrm{BK}}$ can be described explicitly. Choosing a basis and considering the matrix of $\varphi : M \hookrightarrow M[\frac{1}{u}]$ with respect to that basis describes a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of rank } n \\ \text{objects of } \mathrm{Mod}_k^{\mathrm{BK}} \end{array} \right\} \leftrightarrow \mathrm{GL}_n(k((u)))/\sim$$

where $A \sim B$ if there exists $C \in \mathrm{GL}_n(k[[u]])$ such that $A = C^{-1}B\varphi(C)$. Recall that any invertible matrix over $k((u))$ can be written as $C_1\Lambda C_2$ where $\Lambda = \mathrm{diag}(u^{r_i})$ and $C_i \in \mathrm{GL}_n(k[[u]])$. Under this bijection:

- If M is an object of $\mathrm{Mod}_k^{\mathrm{BK}}$ corresponding to a φ-conjugacy class represented by $C_1\Lambda C_2$ then the $(r_i) = \mathrm{Weight}(M)$.
- The isomorphism classes of Breuil–Kisin modules satisfying the equivalent conditions of Lemma 4.3.4 identify with the φ-conjugacy classes represented by matrices $C_1\Lambda$ with $C_1 \in \mathrm{GL}_n(k[[u]])$ and $\Lambda = \mathrm{diag}(u^{r_i})$.

It will not be true that the collection of objects in $\mathrm{Mod}_k^{\mathrm{BK}}$ satisfying the equivalent properties of Lemma 4.3.4 is stable under subquotients. However this will be true if one considers such M with $\mathrm{Weight}(M) \subset [0, p]$.

Definition 4.3.7. An object of $\mathrm{Mod}_k^{\mathrm{BK}}$ is strongly divisible if M satisfies the equivalent properties of Lemma 4.3.4 and if $\mathrm{Weight}(M) \subset [0, p]$. We denote the full subcategory of strongly divisible Breuil–Kisin modules by $\mathrm{Mod}_k^{\mathrm{SD}}$.

Remark 4.3.8. If $M \in \mathrm{Mod}_k^{\mathrm{BK}}$ then there are exact sequences

$$0 \to \mathrm{gr}^{i-1}(M^\varphi) \xrightarrow{u} \mathrm{gr}^i(M^\varphi) \to \mathrm{gr}^i(M_k^\varphi) \to 0$$
$$0 \to \mathrm{gr}^{i-p}(M) \xrightarrow{u} \mathrm{gr}^i(M) \to \mathrm{gr}^i(M_k) \to 0$$

The first is just the exact sequence (4.1.9) in the case $M = M$ and $N = M^\varphi$ with $A = k[[u]]$ and $a = u$. The second exact sequence is obtained similarly (using that $F^i(uM) = u(F^{i-p}M)$).

Lemma 4.3.9. *Let* $0 \to M \to N \to P \to 0$ *be an exact sequence in* $\mathrm{Mod}_k^{\mathrm{BK}}$.

(1) *The map* $N \to P$ *is strict when viewed as a map of filtered modules if and only if* $0 \to M_k \to N_k \to P_k \to 0$ *is an exact sequence in* $\mathrm{Fil}(k)$ *in the sense of Notation 4.1.12.*
(2) *The map* $N^\varphi \to P^\varphi$ *is strict if and only if* $0 \to M_k^\varphi \to N_k^\varphi \to P_k^\varphi \to 0$ *is exact in* $\mathrm{Fil}(k)$
(3) *Statement (2) is equivalent to* $M_k^\varphi \to N_k^\varphi$ *being strict, which is equivalent to* $N_k^\varphi \to P_k^\varphi$ *being strict.*

Proof. Note that $M \to N$ is strict as a map of filtered modules. To see this suppose $m \in M \cap F^iN$, then $\varphi(m) \in \varphi(M) \cap u^iN \subset M[\frac{1}{u}] \cap u^iN$. Since $M \to N$ has u-torsionfree cokernel $M[\frac{1}{u}] \cap u^iN = u^iM$. Thus $m \in F^iM$. Similarly $M^\varphi \to N^\varphi$ is strict. We deduce that $N \to P$ is strict if and only if $0 \to \mathrm{gr}^i(M) \to \mathrm{gr}^i(N) \to \mathrm{gr}^i(P) \to 0$ is exact for each i (Lemma 4.1.6). Likewise $N^\varphi \to P^\varphi$ is strict if and only if $0 \to \mathrm{gr}^i(M^\varphi) \to \mathrm{gr}^i(N^\varphi) \to \mathrm{gr}^i(P^\varphi) \to 0$ is exact.

Using the second exact sequence of Remark 4.3.8 we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{gr}^{i-p}(M) & \xrightarrow{u} & \mathrm{gr}^i(M) & \longrightarrow & \mathrm{gr}^i(M_k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{gr}^{i-p}(N) & \xrightarrow{u} & \mathrm{gr}^i(N) & \longrightarrow & \mathrm{gr}^i(N_k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{gr}^{i-p}(P) & \xrightarrow{u} & \mathrm{gr}^i(P) & \longrightarrow & \mathrm{gr}^i(P_k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The previous paragraph shows that if $N \to P$ is strict then the left and middle columns are exact, and so the right column is exact also. Conversely if the right column is exact then one proves the middle column is exact by increasing induction on i (for small enough i, $F^iM = M$ so $\mathrm{gr}^i(M) = 0$). This proves (1). The same argument but with the diagram replaced with the diagram obtained by considering the first exact sequence of Remark 4.3.8 proves (2) also.

It remains to show that if $M_k^\varphi \to N_k^\varphi$ or $N_k^\varphi \to P_k^\varphi$ is strict then $0 \to M_k^\varphi \to N_k^\varphi \to P_k^\varphi \to 0$ is exact. It suffices to show that $\sum_{i \in \mathrm{Weight}(M)} i + \sum_{i \in \mathrm{Weight}(P)} i = \sum_{i \in \mathrm{Weight}(N)} i$ after Corollary 4.1.13. Remark 4.3.6 says that $\sum_{i \in \mathrm{Weight}(M)} i$ equals the u-adic valuation of the determinant of $\varphi : M \to M[\frac{1}{u}]$. Since this is clearly additive on exact sequences the lemma follows. $\square$

Lemma 4.3.10. *Let $0 \to M \to N \to P \to 0$ be an exact sequence in $\mathrm{Mod}_k^{\mathrm{BK}}$. Suppose M and P satisfy the equivalent conditions of Lemma 4.3.4. If $N \to P$ is strict then N satisfies the equivalent conditions of Lemma 4.3.4 also.*

Proof. Consider the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{gr}^i(M_k^\varphi) & \longrightarrow & \mathrm{gr}^i(N_k^\varphi) & \longrightarrow & \mathrm{gr}^i(P_k^\varphi) & \longrightarrow & 0 \\
 & & \wr\!\uparrow & & \uparrow & & \wr\!\uparrow & & \\
0 & \longrightarrow & \mathrm{gr}^i(M_k) & \longrightarrow & \mathrm{gr}^i(N_k) & \longrightarrow & \mathrm{gr}^i(P_k) & \longrightarrow & 0
\end{array}$$

The left and right vertical arrows are isomorphisms by assumption. Since $N \to P$ is strict, part (1) of Lemma 4.3.9 implies the bottom row is exact. Thus $\mathrm{gr}^i(N_k^\varphi) \to \mathrm{gr}^i(P_k^\varphi)$ is surjective and so $N_k^\varphi \to P_k^\varphi$ is strict by Lemma 4.1.6. Part (3) of Lemma 4.3.9 then implies the top row is exact. We conclude that $N_k \to N_k^\varphi$ is an isomorphism in $\mathrm{Fil}(k)$. $\square$

Lemma 4.3.11. *Let $0 \to M \to N \to P \to 0$ be an exact sequence in $\mathrm{Mod}_k^{\mathrm{BK}}$. Suppose that N satisfies the equivalent conditions of Lemma 4.3.4* *and that $M_k \to N_k$ is strict. Then $N \to P$ is strict and M and P also satisfy the equivalent conditions of Lemma 4.3.4.*

PROOF. The following diagram of objects in $\mathrm{Fil}(k)$ commutes.

$$\begin{array}{ccc} M_k^{\varphi} & \longrightarrow & N_k^{\varphi} \\ \uparrow & & \uparrow \\ M_k & \longrightarrow & N_k \end{array}$$

As maps of k-vector spaces the horizontal arrows are injective and the vertical arrows are isomorphisms. By assumption the maps $M_k \to N_k$ and $N_k \to N_k^{\varphi}$ are strict. It follows that $M_k^{\varphi} \to N_k^{\varphi}$ and $M_k \to M_k^{\varphi}$ are strict also.

The following is also a commutative diagram in $\mathrm{Fil}(k)$.

$$\begin{array}{ccc} N_k^{\varphi} & \longrightarrow & P_k^{\varphi} \\ \uparrow & & \uparrow \\ N_k & \longrightarrow & P_k \end{array}$$

As maps of k-vector spaces the vertical maps are isomorphisms and the horizontal arrows are surjections. By assumption the leftmost vertical arrow is strict. Using part (3) of Lemma 4.3.9, $M_k^{\varphi} \to N_k^{\varphi}$ being strict implies $N_k^{\varphi} \to P_k^{\varphi}$ is strict. It follows that $P_k \to P_k^{\varphi}$ and $N_k \to P_k$ are strict. Thus M and P are as in Lemma 4.3.4 and after (1) of Lemma 4.3.9 we know $N \to P$ is strict. $\square$

LEMMA 4.3.12. *Suppose N is strongly divisible. If $0 \to M \to N \to P \to 0$ is an exact sequence in $\mathrm{Mod}_k^{\mathrm{BK}}$ then $M_k \to N_k$ is strict.*

PROOF. We have a commutative diagram with exact rows (Remark 4.3.8)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{gr}^{i-p}(M) & \longrightarrow & \mathrm{gr}^{i}(M) & \longrightarrow & \mathrm{gr}^{i}(M_k) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow\alpha & & \\ 0 & \longrightarrow & \mathrm{gr}^{i-p}(N) & \longrightarrow & \mathrm{gr}^{i}(N) & \longrightarrow & \mathrm{gr}^{i}(N_k) & \longrightarrow & 0 \end{array}$$

One knows that $M \to N$ is strict (as was shown in the first paragraph of the proof of Lemma 4.3.9) so the left and middle vertical arrows are injective by Lemma 4.1.6. We have to show α is injective for every i.

Let us show that $\mathrm{gr}^i(N) = 0$ for $i < p$. For injectivity of α when $i < p$ we argue as follows. As $\mathrm{Weight}(N) \subset [0,p]$, and because $N_k \cong N_k^{\varphi}$, we have $\mathrm{gr}^i(N_k) = 0$ for $i < 0$. Hence $\mathrm{gr}^i(N) = \mathrm{gr}^{i-p}(N)$ for $i < 0$. This implies $\mathrm{gr}^i(N) = 0$ for $i < 0$ because for small enough i, $F^iN = N$. Using the diagram we deduce that $\mathrm{gr}^i(M) = 0$ for $i < 0$ also, and that for $i < p$ we have $\mathrm{gr}^i(M) = \mathrm{gr}^i(M_k)$ and $\mathrm{gr}^i(N) = \mathrm{gr}^i(N_k)$. This proves α is injective when $i < p$.

For injectivity of α when $i \geq p$ it suffices to show $F^iN_k = 0$ for $i > p$ (because then $F^iM_k = 0$ for $i > p$ so α is just the zero map when $i > p$ and when $i = p$, α is the inclusion $F^iM_k \to F^iN_k$). Let us prove this is the case. Since $\mathrm{Weight}(N) \subset [0,p]$ we have $\mathrm{gr}^i(N_k) = 0$ for $i > p$; it suffices to show $F^i N_k = 0$ for $i >> p$. But N_k is both Hausdorff (being a quotient of N, which is Hausdorff) and a finite dimensional k-vector space, this forces $F^i N_k$ to vanish for large i. So we are done. □

Putting all this together we deduce the following.

PROPOSITION 4.3.13. *Let* $0 \to M \to N \to P \to 0$ *be an exact sequence in* $\mathrm{Mod}_k^{\mathrm{BK}}$.

(1) *If* $N \in \mathrm{Mod}_k^{\mathrm{SD}}$ *then* M *and* P *are strongly divisible and the sequence*
$$0 \to M_k^{\varphi} \to N_k^{\varphi} \to P_k^{\varphi} \to 0$$
is exact in $\mathrm{Fil}(k)$. *Thus* $\mathrm{Weight}(N) = \mathrm{Weight}(M) \cup \mathrm{Weight}(P)$.

(2) *If* $P, M \in \mathrm{Mod}_k^{\mathrm{SD}}$ *then* $N \in \mathrm{Mod}_k^{\mathrm{SD}}$ *if and only if* $N \to P$ *is strict.*

PROOF. This follows by putting together Lemma 4.3.9, Lemma 4.3.10, Lemma 4.3.11 and Lemma 4.3.12. □

4. Strong Divisibility with Coefficients

In this section we adapt the discussion of the previous section to allow for coefficients.

NOTATION 4.4.1. Let $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ denote the full subcategory of $\mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ whose objects are finite free over $k[[u]]$. We say a pair (M, ι) is strongly divisible if M is, and let $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ denote the full subcategory of such objects.

REMARK 4.4.2. As in Remark 2.5.10 any $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ is free as a module over $k[[u]] \otimes_{\mathbb{F}_p} k_E$. We can therefore decompose
$$M = \prod_{\tau : k \hookrightarrow k_E} M_\tau$$
with each M_τ a finite free module over $k_E[[u]]$. Note that the filtration on M is k_E-stable and so likewise M_k and M_k^{φ} decompose into $\prod_\tau M_{k,\tau}$ and $\prod_\tau M_{k,\tau}^{\varphi}$ as filtered vector spaces (each component being a filtered k_E-vector space).

We can refine Definition 4.3.3 for objects M of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$.

DEFINITION 4.4.3. For each $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, k_E)$ let $\mathrm{Weight}_\tau(M)$ be the multiset of integers which contains i with multiplicity equal to
$$\dim_{k_E} \mathrm{gr}^i(M_{k,\tau}^{\varphi})$$
Since $M_k^{\varphi} = \prod M_{k,\tau}^{\varphi}$ we have that $\mathrm{Weight}(M)$ equals the union over all τ of $[k_E : k]$ copies of $\mathrm{Weight}_\tau(M)$.

The following is a version of Lemma 4.3.4 for objects of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ and is proved in exactly the same fashion.

Lemma 4.4.4. *Let M be an object of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$. Then the following are equivalent:*

(1) *The semilinear map $M_k \to M_k^{\varphi}$ is an isomorphism of filtered modules.*
(2) *For each $\tau : k \to k_E$ there exists a $k_E[[u]]$-basis (f_i) of M_τ and integers (r_i) such that $(u^{r_i} f_i)$ is a $k_E[[u^p]]$-basis of $\varphi(M)_\tau$.*

Allowing coefficients Proposition 4.3.13 refines to:

Proposition 4.4.5. *Let $0 \to M \to N \to P \to 0$ be an exact sequence in $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$.*

(1) *If $N \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ then M and P are both strongly divisible and for each $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, k_E)$ we have $\mathrm{Weight}_\tau(N) = \mathrm{Weight}_\tau(M) \cup \mathrm{Weight}_\tau(P)$.*
(2) *If $M, P \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ then $N \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ if and only if $N \to P$ is strict.*

Remark 4.4.6. The analogue of Remark 4.3.6 for $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ says that: choosing $k_E[[u]]$-bases for each M_τ and taking the matrices representing φ with respect to these bases describes a bijection

$$\left\{ \begin{array}{c} \text{isomorphism classes of rank } n \\ \text{objects of } \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O}) \end{array} \right\} \leftrightarrow \mathrm{GL}_n(k_E((u)))^f / \sim$$

where $f = [K : \mathbb{Q}_p]$ and where two f-tuples of matrices satisfy $(A_\tau) \sim (B_\tau)$ if there exist $C_\tau \in \mathrm{GL}_n(k_E[[u]])$ such that $A_\tau = C_\tau^{-1} B_\tau \varphi(C_{\tau\circ\varphi})$ for all τ. As in Remark 4.3.6 each A_τ can be written as $C_\tau \Lambda_\tau C_\tau'$ with $C_\tau, C_\tau' \in \mathrm{GL}_n(k_E[[u]])$ and $\Lambda_\tau = \mathrm{diag}(u^{r_{i,\tau}})$.

- The multiset $\{r_{i,\tau}\}$ is the multiset $\mathrm{Weight}_\tau(M)$.
- The M which satisfy Lemma 4.4.4 correspond to classes represented by an f-tuple of matrices (A_τ) such that each $A_\tau = C_\tau \Lambda_\tau$.

Remark 4.4.7. Just as in Remark 4.3.5 if bases as in (2) of Lemma 4.4.4 exist then the multiset $\{r_{i,\tau}\}$ is characterised by the property that it contains i with multiplicity equal to the k_E-dimension of $\mathrm{gr}^i(M_{k,\tau})$. In particular the multiset is equal to $\mathrm{Weight}_\tau(M)$.

5. Crystalline Representations and Strong Divisibility

In this brief section we provide the motivation for the category $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$.

Theorem 4.5.1 (Gee–Liu–Savitt). *Assume $K = K_0$ and that $K_\infty \cap K(\mu_{p^\infty}) = K$. Let T be a crystalline $\mathcal{O}$-lattice with $\mathrm{HT}_\tau(T) \subset [0,p]$ for each $\tau : k \to k_E$. Then $M = M(T)/\varpi$ is an object of $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ and $\mathrm{Weight}_\tau(M) = \mathrm{HT}_\tau(T)$.*

Proof. In the case $p > 2$ this follows from [**16**, Theorem 4.22]. We give a proof without the $p = 2$ assumption, see Theorem 9.4.14 with all $n_\tau = 1$. □

Remark 4.5.2. If $p > 2$ then it is always true that $K_\infty \cap K(\mu_{p^\infty}) = K$. When $p = 2$ is it possible that $K_\infty \cap K(\mu_{p^\infty}) \neq K$. However Wang [**29**, Lemma 2.1] has shown that it is always possible to choose the uniformiser π such that $K_\infty \cap K(\mu_{p^\infty}) = K$.

6. Rank Ones

In this section we recall a standard classification of the rank one objects $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$. Further we describe the character through which G_{K_∞} acts on $T(M)$ in terms of the fundamental characters for K. For the convenience of the reader we repeat the definition of these characters given in the introduction.

Definition 4.6.1. Let π_K be a $p^f - 1$-th root of π. For each $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, k_E)$ the τ-th fundamental character of K is defined to be the composite

$$\chi_\tau : G_K \to \mathcal{O}_{K(\pi_K)}^\times \to k^\times \xrightarrow{\tau} k_E^\times$$

where the first map is given by $\sigma \mapsto \frac{\sigma(\pi_K)}{\pi_K}$. This character does not depend upon the choice of π_K. Note also that $\chi_{\tau\circ\varphi} = \chi_\tau^p$.

Notation 4.6.2. If $M \in \mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ is free of rank one over $k[[u]] \otimes_{\mathbb{F}_p} k_E$ then each M_τ is free of rank one over $k_E[[u]]$; choosing generators e_θ we find $\varphi(e_{\tau\circ\varphi}) = A_\tau e_\tau$ for some $A_\tau \in k_E((u))$. By changing basis we can arrange (see [**16**, Lemma 6.2]) that for any fixed τ' we have

$$A_\tau = (a)_\tau u^{r_\tau}$$

where $r_\tau \in \mathbb{Z}$, $a \in k_E^\times$ and $(a)_\tau = a$ if $\tau = \tau'$ and 1 otherwise. In this case we write $M \cong \overline{\mathfrak{S}}(\{r_\tau\}; a)$.

Remark 4.6.3. Any rank one object of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ satisfies the equivalent conditions of Lemma 4.4.4. Since $\mathrm{Weight}_\tau(\overline{\mathfrak{S}}(\{r_\tau\}; a)) = \{r_\tau\}$ we have that $\overline{\mathfrak{S}}(\{r_\tau\}; a) \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ if $r_\tau \in [0, p]$.

Lemma 4.6.4. *The action of G_{K_∞} on $T(\overline{\mathfrak{S}}(\{r_\tau\}; a))$ is through the (restriction to G_{K_∞} of the) character*

$$\psi_a \prod_{\tau : k \to k_E} \chi_\tau^{-r_\tau}$$

where ψ_a denotes the unramified character sending geometric frobenius onto a.

Proof. It is clear that $\overline{\mathfrak{S}}(\{r_\tau\}; a) \otimes \overline{\mathfrak{S}}(\{s_\tau\}; b) = \overline{\mathfrak{S}}(\{r_\tau + s_\tau\}; ab)$ so, since $M \mapsto T(M)$ is a tensor functor, it suffices to show that

(1) $T(\overline{\mathfrak{S}}(\{0\}; a)) \cong \psi_a$
(2) $T(\overline{\mathfrak{S}}(\{\delta_{\tau,\tau'}\}; 1) \cong \chi_{\tau'}$

where the $\delta_{\tau,\tau'} = 0$ unless $\tau = \tau'$ in which case it equals -1.

For (2) note that if $x \in C^\flat$ satisfies $\varphi^f(x) = ux$ then $\sum e_{\tau'\circ\varphi^i} \otimes \varphi^{f-i}(x)$ is an element of $(\overline{\mathfrak{S}}(\{\delta_{\tau,\tau'}\};1) \otimes_{\mathbb{F}_p} C^\flat)^{\varphi=1}$. The set of such x is a one-dimensional k-vector space and so we get a non-zero (and therefore bijective) k_E-linear G_{K_∞}-equivariant map

$$\{x \in C^\flat \mid \varphi^f(x) = ux\} \otimes_{k,\tau'} k_E \to T((\{\delta_{\tau,\tau'}\};1))$$

The map $\{x \in C^\flat \mid \varphi^f(x) = ux\} \to \{z \in C \mid z^{p^f} = \pi z\}$ given by $x \mapsto x^\sharp$ is bijective and G_{K_∞}-equivariant. As the action of G_{K_∞} on $\{z \in C \mid z^{p^f} = \pi z\}$ is through the composite

$$G_K \to \mathcal{O}^\times_{K(\pi_K)} \to k^\times$$

(the first map being given by $\sigma \mapsto \frac{\sigma(\pi_K)}{\pi_K}$) it follows that G_{K_∞} acts on $T((\{\delta_{\tau,\tau'}\};1))$ through $\chi_{\tau'}$. Isomorphism (1) follows similarly since the G_{K_∞} (even G_K) action on the set of $x \in \overline{k} \subset C^\flat$ satisfying $\varphi^f(x) = a^{-1}x$ is through ψ_a. $\square$

COROLLARY 4.6.5. *We have $T(\overline{\mathfrak{S}}(\{r_\tau\};a)) \cong T(\overline{\mathfrak{S}}(\{s_\tau\};b))$ if and only if $a = b$ and there exists integers $\delta_\tau \in \mathbb{Z}$ such that $r_\tau = p\delta_{\tau\circ\varphi} - \delta_\tau + s_\tau$. Equivalently $a = b$ and for any fixed τ*

$$\sum_0^{f-1} r_{\tau\circ\varphi^i} p^i \equiv \sum_0^{f-1} s_{\tau\circ\varphi^i} p^i \quad \text{modulo } p^f - 1$$

PROOF. This follows because $\chi_{\tau\circ\varphi} = \chi_\tau^p$. $\square$

PROPOSITION 4.6.6. *Let T be a rank one crystalline $\mathcal{O}$-lattice. If $\mathrm{HT}(T) = \{r_\tau\}$ then there exists $a \in k_E^\times$ such that $M(T)/\varpi \cong \overline{\mathfrak{S}}(\{r_\tau\};a)$.*

PROOF. See e.g. [**16**, Lemma 6.3]. $\square$

7. Extensions

The aim of this section is to compute the dimensions of the first Yoneda extension group in the exact category[1] $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$.

[1]The categories $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ and $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ are not abelian categories because cokernels may not be free $k[[u]]$-modules. However both are exact categories. Checking this amounts to checking that either category is closed under forming the pushout of a morphism $f : M \to N$ with torsion-free cokernel along an arbitrary morphism $g : M \to P$. The pushout is constructed as

$$\mathrm{coker}(A \xrightarrow{(-f,g)} M \oplus P)$$

so this is clear for $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$, and follows for $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ from Proposition 4.4.5. We mention this because the formalism of an exact category is sufficient to make the usual construction of the Yoneda extension groups work.

CONSTRUCTION 4.7.1. If M is an object of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ we let $H^0(M)$ and $H^1(M)$ denote the cohomology of the complex

$$M \xrightarrow{\varphi-1} M[\tfrac{1}{u}]$$

The $H^i(M)$ are k_E-vector spaces. If P and M are objects of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ then $H^1(\mathrm{Hom}(P,M)^{\mathcal{O}})$ (see Notation 2.5.9 for the construction of $\mathrm{Hom}(P,M)^{\mathcal{O}}$) can be identified functorially with first Yoneda extension group $\mathrm{Ext}^1_{k_E}(P,M)$ in the exact category $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ as we now explain.

- Write $R = k[[u]] \otimes_{\mathbb{F}_p} k_E$. For any $f \in \mathrm{Hom}_{R[\frac{1}{u}]}(P \otimes_\varphi R[\frac{1}{u}], M[\frac{1}{u}])$ let N_f be the object of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ with underlying R-module $P \bigoplus M$ and frobenius $\varphi_{N_f} = (\varphi_M + f, \varphi_P)$. Then N_f sits in an exact sequence
$$0 \to M \to N_f \to P \to 0$$
One checks that this construction induces a homomorphism of abelian groups $\mathrm{Hom}(P \otimes_\varphi R[\frac{1}{u}], M[\frac{1}{u}]) \to \mathrm{Ext}^1_{k_E}(P,M)$ which is functorial in P and M. In particular it is a homomorphism of k_E-vector spaces. Since every extension in $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ of P by M splits as an R-module this homomorphism is surjective.
- To compute the kernel of this homomorphism suppose we have a commutative diagram in $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ as below.
$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & N_f & \longrightarrow & P & \longrightarrow & 0 \\ & & \| & & \downarrow\wr & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & M \bigoplus P & \longrightarrow & P & \longrightarrow & 0 \end{array}$$
The isomorphism $N_f \xrightarrow{\sim} M \oplus P$ may be written as $(\mathrm{Id}_M + \beta, \mathrm{Id}_P)$ for some $\beta \in \mathrm{Hom}_R(P,M)$. Unwinding what it means for this to be an isomorphism in $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$, we see that
$$f + \beta[\tfrac{1}{u}] \circ \varphi_P = \varphi_M \circ \varphi^*\beta \tag{4.7.2}$$
where $\beta[\frac{1}{u}] = \beta \otimes \mathrm{Id} : P[\frac{1}{u}] \to M[\frac{1}{u}]$ and $\varphi^*\beta = \beta \otimes \mathrm{Id} : P \otimes_\varphi k((u)) \to M \otimes_\varphi k((u))$.
- We can identify $\mathrm{Hom}_{R[\frac{1}{u}]}(P \otimes_\varphi R[\frac{1}{u}], M[\frac{1}{u}])$ with $\mathrm{Hom}_{R[\frac{1}{u}]}(P[\frac{1}{u}], M[\frac{1}{u}]) = \mathrm{Hom}(P,M)^{\mathcal{O}}[\frac{1}{u}]$ via φ_P^{-1}. Under this identification (4.7.2) shows that the kernel of $\mathrm{Hom}(P,M)^{\mathcal{O}}[\frac{1}{u}] \to \mathrm{Ext}^1_{k_E}(P,M)$ is equal to the image under $\varphi - 1$ of $\mathrm{Hom}(P,M)^{\mathcal{O}}$. In other words there are functorial identifications
$$H^1(\mathrm{Hom}(P,M)^{\mathcal{O}}) = \mathrm{Ext}^1_{k_E}(P,M) \tag{4.7.3}$$

EXAMPLE 4.7.4. The space of extensions $\mathrm{Ext}^1_{k_E}(P,M)$ is infinite dimensional and too large. The following example describes an extension which we do not wish to consider. Let $d \geq 1$.

$$N = k[[u]]^2, \qquad \varphi = \begin{pmatrix} u & 1 \\ 0 & u^d \end{pmatrix}$$

On the one hand[2] $\mathrm{Weight}(N) = \{0, d+1\}$. On the other hand N is an extension of two rank one Breuil–Kisin modules with weights 1 and d. We shall avoid this phenomenon by describing particular subsets of $\mathrm{Ext}^1_{\mathrm{k_E}}$.

VARIANT 4.7.5 (Effective extensions). The following construction gives a subset of $\mathrm{Ext}^1_{k_E}$ for which the weights behave well. If $M \in \mathrm{Mod}^{\mathrm{BK}}_k(\mathcal{O})$ let $H^i_{\mathrm{eff}}(M)$ denote the cohomology of the complex

$$\varphi - 1 : F^0 M^\varphi \to M^\varphi$$

Then $H^0_{\mathrm{eff}}(M) = H^0(M)$. The inclusion $M^\varphi \to M[\frac{1}{u}]$ induces a map $H^1_{\mathrm{eff}}(M) \to H^1(M)$ which is injective (if $m \in M^\varphi$ can be written as $\varphi(m') - m'$ with $m' \in M$ then $m' \in M \cap M^\varphi$). Define

$$\mathrm{Ext}^1_{\mathrm{eff}}(P, M) \subset \mathrm{Ext}^1_{k_E}(P, M)$$

to be the image of $H^1_{\mathrm{eff}}(\mathrm{Hom}(P,M)^{\mathcal{O}})$ under (4.7.3). We remark that $\mathrm{Ext}^1_{\mathrm{eff}}$ is finite dimensional (we shall not give a proof). We shall not use $\mathrm{Ext}^1_{\mathrm{eff}}$ in the following (instead we refine this construction in Variant 4.7.9) but feel it is worth introducing to illustrate the ideas of this section. For this reason we give the proof of the following lemma (though it too will not be used later on).

LEMMA 4.7.6. *Let* $0 \to M \to N \to P \to 0$ *be an exact sequence in* $\mathrm{Mod}^{\mathrm{BK}}_k(\mathcal{O})$. *Then* $0 \to M^\varphi_k \to N^\varphi_k \to P^\varphi_k \to 0$ *is an exact sequence of filtered modules if and only if* N *represents a class in* $\mathrm{Ext}^1_{\mathrm{eff}}(P, M)$.

PROOF. With notation as in Construction 4.7.1 we can assume $N = N_{f \circ \varphi_P}$ for some $f \in \mathrm{Hom}(P,M)^{\mathcal{O}}[\frac{1}{u}]$. Thus as a module $N = M \oplus P$ with $\varphi_N = (\varphi_M + f \circ \varphi_P, \varphi_P)$.

After Lemma 4.3.9 exactness of $0 \to M^\varphi_k \to N^\varphi_k \to P^\varphi_k \to 0$ is equivalent to $N^\varphi \to P^\varphi$ being strict. Note that if $z' \in P_{\tau\circ\varphi} \otimes_{k_E[[u]],\varphi} k_E[[u]]$ is such that $\varphi_P(z') = z$ then

$$\varphi(0, z') = (f(z), z)$$

Thus strictness of $N^\varphi \to P^\varphi$ is equivalent to asking that for each $z \in F^i P^\varphi_\tau$ there exists $m \in u^i M_\tau$ such that $m - f(z) \in M^\varphi_\tau$. Since $f \in \mathrm{Hom}(P,M)^{\mathcal{O},\varphi}$ if and only if $f(P^\varphi) \subset M^\varphi$ we immediately see that $f \in \mathrm{Hom}(P,M)^{\mathcal{O},\varphi}$ implies $N^\varphi \to P^\varphi$ is strict, as we can then take $m = 0$.

Now suppose $N^\varphi \to P^\varphi$ is strict. We must find $g \in \mathrm{Hom}(P,M)^{\mathcal{O}}$ such that $f - g + \varphi(g) \in \mathrm{Hom}(P,M)^{\mathcal{O},\varphi}$. For each τ we can choose a basis z_i of P_τ such that $u^{r_i} z_i$ form a basis of P^φ. Applying the above paragraph we obtain $m_i \in u^{r_i} M$ such that $m_i - f(u^{r_i} z_i) \in M^\varphi$. Let $g \in \mathrm{Hom}(P,M)^{\mathcal{O}}$ be the function which on P_τ is given by $z_i \mapsto \frac{m_i}{u^{r_i}}$. Then $f - g + \varphi(g)$ acts on P^φ_τ by:

$$u^{r_i} z \mapsto f(u^{r_i} z) - m_i + \varphi(g)(u^{r_i} z) \in M^\varphi_\tau$$

Thus $f - g + \varphi(g) \in \mathrm{Hom}(P,M)^{\mathcal{O},\varphi}$. ∎

[2]Because $\begin{pmatrix} u & 1 \\ 0 & u^d \end{pmatrix} = \begin{pmatrix} 1+u & 0 \\ u^d & (1+u)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u^{d+1} \end{pmatrix} \begin{pmatrix} u(1+u)^{-1} & (1+u)^{-1} \\ -1 & 1 \end{pmatrix}$

Variant 4.7.7. A possible variant of $\mathrm{Ext}^1_{\mathrm{eff}}$ is described as follows. We will briefly consider such extensions in the following chapter. If $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ let $H^i_{[0,p]}(M)$ denote the cohomology of the complex

$$F^{-p}M \xrightarrow{\varphi-1} u^{-p}M$$

Again $H^0_{[0,p]}(M) = H^0(M)$ and $H^1_{[0,p]}(M) \subset H^1(M)$. Note that if all the weights of M are $\geq -p$ then $M^\varphi \subset u^{-p}M$ and so $H^1_{\mathrm{eff}}(M) \subset H^1_{[0,p]}(M)$. Define

$$\mathrm{Ext}^1_{[0,p]}(P, M) \subset \mathrm{Ext}^1_{k_E}(P, M)$$

to be the image of $H^1_{[0,p]}(\mathrm{Hom}(P, M)^{\mathcal{O}})$ under (4.7.3). The relevance of this definition is explained by the following lemma.

Lemma 4.7.8. *Let* $0 \to M \to N \to P \to 0$ *be an exact sequence in* $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$. *Suppose* $\mathrm{Weight}(N) \subset [0, p]$. *Then* $\mathrm{Weight}(P)$ *and* $\mathrm{Weight}(M) \subset [0, p]$ *and* N *represents a class in* $\mathrm{Ext}^1_{[0,p]}(P, M)$.

Proof. Note that $\mathrm{Weight}(P) \subset [0, p]$ is equivalent to asking that $u^pP \subset P^\varphi \subset P$ (and likewise with P replaced with N or M). Since $0 \to M \to N \to P \to 0$ is exact with each term a free module we see that both M and P have weights contained in $[0, p]$.

It remains to show that N represents a class in $\mathrm{Ext}^1_{[0,p]}(P, M)$. We may suppose that $N = N_{f\circ\varphi_P}$ for some $f \in \mathrm{Hom}(P, M)^{\mathcal{O}}[\frac{1}{u}]$ as in Construction 4.7.1. Thus $N = M \oplus P$ with $\varphi_N = (\varphi_M + f \circ \varphi_P, \varphi_P)$. Suppose $f \notin u^{-p}\,\mathrm{Hom}(P, M)^{\mathcal{O}}$, so that $f(u^pP) \not\subset M$. Since $u^pP \subset P^\varphi$ there is a $p' \in P \otimes_{k[[u]],\varphi} k[[u]]$ such that $\varphi_P(p') \in u^pP$ and $f \circ \varphi_P(p') \notin M$. However then $\varphi(0, p') = (f \circ \varphi_P(p'), \varphi_P(p'))$ which is not contained in N. $\square$

Variant 4.7.9 (Strongly Divisible Extensions). The following refinement of Variant 4.7.5 will be the focus of our attention for the rest of this section. If M is an object of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ then we let $H^i_{\mathrm{SD}}(M)$ denote the cohomology of the complex

$$M \cap \varphi(M) \xrightarrow{\varphi-1} \varphi(M)$$

Then $H^0_{\mathrm{SD}}(M) = H^0(M)$. The inclusion $\varphi(M) \to M[\frac{1}{u}]$ induces a map $H^1_{\mathrm{SD}}(M) \to H^1(M)$ which is injective (if $m \in \varphi(M)$ can be written as $\varphi(m') - m'$ with $m' \in M$ then $m' \in M \cap \varphi(M)$). Let

$$\mathrm{Ext}^1_{\mathrm{SD}}(P, M) \subset \mathrm{Ext}^1_{k_E}(P, M)$$

denote the image of $H^1_{\mathrm{SD}}(\mathrm{Hom(P,M)}^{\mathcal{O}})$ under (4.7.3). Note that $\mathrm{Ext}^1_{\mathrm{SD}}(P, M) \subset \mathrm{Ext}^1_{\mathrm{eff}}(P, M)$.

Remark 4.7.10. Note that for $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$, the cokernel of $\varphi - 1 : F^0M \to M$ injects into $H^1(M)$ (the map sends an element in the cokernel represented by $m \in M$ onto the class in $H^1(M)$ represented by m; if this latter class is zero then $m = \varphi(n) - n$ for some $n \in N$ and so $n \in F^0M$, i.e.

m is zero in the cokernel). The image of this injection coincides with image of $H^1_{\mathrm{SD}}(M) \subset H^1(M)$ since if $f \in M$ then $f \equiv \varphi(f)$ modulo $(\varphi - 1)(M)$. This means that if

$$0 \to M \to N \to P \to 0$$

is an exact sequence representing a class in $\mathrm{Ext}^1_{\mathrm{SD}}(P, M)$ then, with notation as in Construction 4.7.1, we can identify $N = N_{f \circ \varphi_P}$ with $f \in \mathrm{Hom}(P, M)^{\mathcal{O}}$ or with $f \in \varphi(\mathrm{Hom}(P, M)^{\mathcal{O}})$, as we choose.

CONSTRUCTION 4.7.11. If $0 \to M \to N \to P \to 0$ is an exact sequence in $\mathrm{Mod}^{\mathrm{BK}}_k(\mathcal{O})$ and $Q \in \mathrm{Mod}^{\mathrm{BK}}_k(\mathcal{O})$ then tensoring (in the sense of Remark 2.5.9) with Q induces a map

$$\mathrm{Ext}^1_{k_E}(P, M) \to \mathrm{Ext}^1_{k_E}(P \otimes Q, M \otimes Q)$$

Under (4.7.3) this corresponds to the map induced by $f \mapsto f \otimes \mathrm{Id}$ on $\mathrm{Hom}(P, M)^{\mathcal{O}} \to \mathrm{Hom}(P \otimes Q, M \otimes Q)^{\mathcal{O}}$. As $f \mapsto f \otimes \mathrm{Id}$ is φ-equivariant we see that $\mathrm{Ext}^1_{\mathrm{SD}}(P, M)$ is mapped into $\mathrm{Ext}^1_{\mathrm{SD}}(P \otimes Q, M \otimes Q)$.

LEMMA 4.7.12. *Let* $0 \to M \to N \to P \to 0$ *be an extension in* $\mathrm{Mod}^{\mathrm{BK}}_k(\mathcal{O})$ *and suppose that* P *satisfies the equivalent conditions of Lemma 4.4.4. Then* $N \to P$ *is strict if and only if, under the identification* (4.7.3)*, the class of this extension lies in* $\mathrm{Ext}^1_{\mathrm{SD}}(P, M)$*.*

PROOF. The proof is similar to that of Lemma 4.7.6. With notation as in Construction 4.7.1 we can assume $N_{f \circ \varphi_P}$ for some $f \in \mathrm{Hom}(P, M)^{\mathcal{O}}[\frac{1}{u}]$. Thus as a module $N = M \bigoplus P$ and $\varphi_N = (\varphi_M + f \circ \varphi_P, \varphi_P)$. We have to show $N \to P$ being strict implies there exists $g \in \mathrm{Hom}(P, M)^{\mathcal{O}}$ such that $f - g + \varphi(g) \in \varphi(\mathrm{Hom}(P, M)^{\mathcal{O}})$, and conversely if $f \in \varphi(\mathrm{Hom}(P, M)^{\mathcal{O}})$ then $N \to P$ is strict. Note that:

- If $f \in \mathrm{Hom}(P, M)^{\mathcal{O}}[\frac{1}{u}]$ then $f \in \varphi(\mathrm{Hom}(P, M)^{\mathcal{O}})$ if and only if $f(\varphi(P)) \subset \varphi(M)$.

The map $N \to P$ is strict if and only if, for every τ and every $z \in F^i P_{\tau \circ \varphi}$, there exists $(m', z) \in N_{\tau \circ \varphi}$ such that

$$\varphi((m', z)) = (\varphi(m') + f(\varphi(z)), \varphi(z)) \in u^i N_\tau$$

Equivalently there exists $m \in u^i M_\tau$ such that $f(\varphi(z)) - m \in \varphi(M)_\tau$. Thus if $f \in \varphi(\mathrm{Hom}(P, M)^{\mathcal{O}})$ then it is immediate that $N \to P$ is strict (take $m = 0$).

As P satisfies the equivalent condition of Lemma 4.4.4 we can find, for each τ, a basis (z_i) of P_τ and integers r_i such that $u^{r_i} z_i$ forms a basis of $\varphi(P)_\tau$ (Lemma 4.4.4). If $N \to P$ is strict then we may choose $m_i \in u^{r_i} M_\tau$ such that $f(u^{r_i} z_i) - m_i \in \varphi(M)_\tau$. Define $g \in \mathrm{Hom}(P, M)^{\mathcal{O}}$ by asserting that the function g acts on P_τ by $z_i \mapsto u^{-r_i} m_i$. Then $f - g + \varphi(g)$ acts on $\varphi(P)_\tau$ by

$$u^{r_i} z_i \mapsto f(u^{r_i} z_i) - m_i + \varphi_M \circ g \circ \varphi_P^{-1}(u^{r_i} z_i) \in \varphi(M)_\tau$$

Thus $f - g + \varphi(g) \in \varphi(\mathrm{Hom}(P, M)^{\mathcal{O}})$. $\square$

Corollary 4.7.13. *If $P, M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ and $0 \to M \to N \to P \to 0$ is an exact sequence in $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ then N is strongly divisible if and only if this extension represents a class in $\mathrm{Ext}^1_{\mathrm{SD}}(P, M)$.*

Our next aim is to compute the dimension of H^1_{SD}.

Lemma 4.7.14. *Let M be an object of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$. Both $H^1_{\mathrm{SD}}(M)$ and $H^0(M)$ are finite and if $\chi(M) = \dim_{k_E} H^1_{\mathrm{SD}}(M) - \dim_{k_E} H^0(M)$ then*

$$\chi(M) - \chi(uM) = \sum_{i \notin p\mathbb{Z}_{\leq 0} \cup \mathbb{Z}_{\geq 0}} \dim_{k_E} \mathrm{gr}^i(M_k)$$

Proof. Finiteness of $H^0(M)$ follows from Proposition 2.4.6 because $H^0(M) \subset T(M)$. For the rest of the lemma consider the inclusion $uM \to M$. It induces a commutative diagram whose rows are exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^0(uM) & \longrightarrow & F^0M & \longrightarrow & Q_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi-1} & & \downarrow{\scriptstyle \varphi-1} & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & uM & \longrightarrow & M & \longrightarrow & M_k & \longrightarrow & 0
\end{array}$$

The snake lemma yields a long exact sequence (note here we are identifying $H^1_{\mathrm{SD}}(M)$ with the cokernel of $\varphi - 1 : F^0M \to M$ as in Remark 4.7.10)

$$0 \to H^0(uM) \to H^0(M) \to \ker\alpha \to H^1_{\mathrm{SD}}(uM) \to H^1_{\mathrm{SD}}(M) \to \operatorname{coker}\alpha \to 0$$

We claim that non-canonically as a k_E-vector space

$$Q_1 = \bigoplus_{i \in p\mathbb{Z}_{\leq 0} \cup \mathbb{Z}_{\geq 1}} \mathrm{gr}^i(M_k)$$

To see this choose a splitting (as k_E-vector spaces) of the exact sequence $0 \to F^1M \to F^0M \to \mathrm{gr}^0(M) \to 0$. Then we can write $F^0M = F^1M \oplus \mathrm{gr}^0(M)$. Observe that $F^0(uM) = (uM) \cap F^1M$ and that this is the kernel of $F^1M \to F^1M_k$. Therefore

$$F^0M/F^0(uM) = F^1M_k \oplus \mathrm{gr}^0(M)$$

Choosing splittings of $0 \to F^{i+1}M_k \to F^iM_k \to \mathrm{gr}^i(M_k) \to 0$ allows us to identify the first term of the above sum with $\bigoplus_{i \in \mathbb{Z}_{\geq 1}} \mathrm{gr}^i(M_k)$. For the second term: there are exact sequences $0 \to \mathrm{gr}^{i-p}(M) \to \mathrm{gr}^i(M) \to \mathrm{gr}^i(M_k) \to 0$ (Remark 4.3.8); choosing splittings shows that $\mathrm{gr}^0(M) = \bigoplus_{i \in p\mathbb{Z}_{\leq 0}} \mathrm{gr}^i(M_k)$ which verifies the claim.

Provided we have finiteness of the $H^1_{\mathrm{SD}}(uM)$ and $H^1_{\mathrm{SD}}(M)$ the formula of the lemma follows by considering the alternating sums of the dimensions in the long exact sequence above: we see that $\chi(N) - \chi(uN) = \dim_{k_E} \operatorname{coker}\alpha - \dim_{k_E} \ker\alpha$, which is equal to the k_E-dimension of Q_2 minus the k_E-dimension of Q_1. The explicit descriptions of Q_1 and Q_2 just given shows this is equal to

$$\sum_\tau \mathrm{Card}(\{i < 0 \mid i \in \mathrm{Weight}_\tau(M) \text{ and } i \notin p\mathbb{Z}\})$$

To finish the proof we must show that $H^1_{\mathrm{SD}}(M)$ is finite.

Observe that, except for the $H^1_{\mathrm{SD}}(M)$ and $H^1_{\mathrm{SD}}(uM)$, all the terms in the long exact sequence above are finite. Therefore finiteness of $H^1_{\mathrm{SD}}(M)$ can be deduced from finiteness of $H^1_{\mathrm{SD}}(u^nM)$ for large enough n. In fact $H^1_{\mathrm{SD}}(u^nM)$ will vanish for n large enough, as we now show. If n is large $N = u^nM$ satisfies $F^0N = N$ and so $H^1_{\mathrm{SD}}(N) = 0$ if $\varphi - 1$ is surjective as a map $N \cap \varphi(N) = \varphi(N) \to \varphi(N)$. Equivalently $\varphi - 1$ is surjective as a map $N \to N$. Increasing n if necessary we can assume that $F^1N = N$; then for $x \in N$, $\varphi(x) \in uN$ and so the sum $\sum \varphi^i(-x)$ converges to $y \in N$. Since $\varphi(y) - y = x$ it follows that $H^1_{\mathrm{SD}}(N) = 0$ which completes the proof. $\square$

Corollary 4.7.15. *Let M be an object of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ and assume that $\mathrm{gr}^i(M_k) = 0$ for $i < -p$. Then*

$$\chi(M) = \sum_{i \le 0} \dim_{k_E} \mathrm{gr}^i(M_k)$$

Proof. We make the following two observations.

- If $F^1M = M$ (i.e. if $\mathrm{gr}^i(M_k) = 0$ for $i < 1$) then clearly $H^0(M) = 0$. This combined with the observation made in the last paragraph of the proof of Lemma 4.7.14 implies that for any M and sufficiently large n, $\chi(u^nM) = 0$.
- Also note that if $m \in F^i(uM)$ then $\varphi(m) \in u^i(uM)$ and so $u^{-1}m \in F^{i+1-p}M$, and vice-versa. Thus $F^i(uM) = uF^{i+1-p}M$ and so
$$\dim_{k_E} \mathrm{gr}^i((uM)_k) = \dim_{k_E} \mathrm{gr}^{i+1-p}(M_k)$$

Applying Lemma 4.7.14 and using these two observations we see that (even without the condition that $\mathrm{gr}^i(M_k) = 0$ for $i < -p$)

$$\chi(M) = \sum_{n \ge 0} \Big(\sum_{i \notin p\mathbb{Z}_{\le 0} \cup \mathbb{Z}_{\ge 0}} \dim_{k_E} \mathrm{gr}^{i+n(1-p)}(M_k) \Big)$$

Since $\mathrm{gr}^i(M_k) = 0$ for $i < -p$ the inner sum for $n = 0$ counts the dimensions of $\mathrm{gr}^i(M_k)$ for $i < 0$ and $\neq -p$ and the inner sum for $n = 1$ counts the dimension of $\mathrm{gr}^{-p}(M_k)$. The remaining inner sums are all zero which proves the result. $\square$

Proposition 4.7.16. *If P and M are objects of $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ then*

$$\dim_{k_E} \mathrm{Ext}^1_{\mathrm{SD}}(P, M) - \dim_{k_E} \mathrm{Hom}_{\mathrm{BK}}(P, M) = \sum_{\tau} \mathrm{Card}(\{i - j < 0 \mid i \in \mathrm{Weight}_\tau(M), j \in \mathrm{Weight}_\tau(P)\})$$

Proof. First let us show that

$$\{i - j \mid i \in \mathrm{Weight}_\tau(M), j \in \mathrm{Weight}_\tau(P)\} = \mathrm{Weight}_\tau(\mathrm{Hom}(P, M)^{\mathcal{O}})$$

To see this choose a basis (m_i) of M_τ such that $(u^{r_i}m_i)$ is a basis of $\varphi(M)_\tau$. Then the integers r_i are the elements of $\mathrm{Weight}_\tau(M)$. Likewise choose a basis (p_j) of P_τ such that $(u^{s_j}p_j)$ are a basis of $\varphi(P)_\tau$. One checks that if

f_{ij} is the element of $\mathrm{Hom}(P,M)^{\mathcal{O}}$ which is zero everywhere except that it maps $p_j \mapsto m_i$ then the f_{ij} form a basis of $\mathrm{Hom}(P,M)^{\mathcal{O}}_\tau$ and $u^{r_i - s_j} f_{ij}$ forms a basis of $\varphi(\mathrm{Hom}(P,M)^{\mathcal{O}})_\tau$. Now appeal to Remark 4.4.7.

The previous paragraph shows that $\mathrm{Hom}(P,M)^{\mathcal{O}}$ satisfies the equivalent conditions of Lemma 4.4.4 and so $\dim_{k_E} \mathrm{gr}^i(\mathrm{Hom}(P,M)^{\mathcal{O}}_k) = \dim_{k_E} \mathrm{gr}^i(\mathrm{Hom}(P,M)^{\mathcal{O},\varphi}_k)$. Since $\mathrm{Weight}(\mathrm{Hom}(P,M)^{\mathcal{O}}) \subset [-p,p]$ it follows that $\mathrm{gr}^i(\mathrm{Hom}(P,M)^{\mathcal{O}}_k) = 0$ for $i < -p$. Thus Corollary 4.7.15 applies with $M = \mathrm{Hom}(P,M)^{\mathcal{O}}$. Using Construction 4.7.1 to identify $\mathrm{Ext}^1_{\mathrm{SD}}(P,M)$ and $H^1_{\mathrm{SD}}(\mathrm{Hom}(P,M))$ the result follows. $\square$

Remark 4.7.17. This proposition should be compared with the number of possible extensions described in [**16**, Theorem 7.9].

Remark 4.7.18. Let $M, P \in \mathrm{Mod}^{\mathrm{SD}}_k(\mathcal{O})$, and let $0 \to M_1 \to M \to M/M_1 \to 0$ be an exact sequence in $\mathrm{Mod}^{\mathrm{SD}}_k(\mathcal{O})$. Then we have an exact sequence in $\mathrm{Mod}^{\mathrm{BK}}_k(\mathcal{O})$

$$0 \to \mathrm{Hom}(P,M_1)^{\mathcal{O}} \to \mathrm{Hom}(P,M)^{\mathcal{O}} \to \mathrm{Hom}(P,M/M_1)^{\mathcal{O}} \to 0 \tag{4.7.19}$$

We claim this sequence stays exact after applying F^0. After the proof of Lemma 4.3.9 this will follow if $\mathrm{Hom}(P,M)^{\mathcal{O}} \to \mathrm{Hom}(P,M/M_1)^{\mathcal{O}}$ is strict as a map of filtered modules. After Lemma 4.7.12 this will follow if the above exact sequence represents a class in $\mathrm{Ext}^1_{\mathrm{SD}}(\mathrm{Hom}(P,M/M_1)^{\mathcal{O}}, \mathrm{Hom}(P,M_1)^{\mathcal{O}})$ (as in the proof of Proposition 4.7.16 we have that $\mathrm{Hom}(P,M/M_1)^{\mathcal{O}}$ satisfies the equivalent conditions of Lemma 4.4.4). Since the (4.7.19) is obtained from $0 \to M_1 \to M \to M/M_1 \to 0$ by tensoring with $P^\vee = \mathrm{Hom}(P, k[[u]] \otimes_{\mathbb{F}_p} k_E)^{\mathcal{O}}$ this follows from Construction 4.7.11. Thus we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^0\,\mathrm{Hom}(P,M_1)^{\mathcal{O}} & \longrightarrow & F^0\,\mathrm{Hom}(P,M)^{\mathcal{O}} & \longrightarrow & F^0\,\mathrm{Hom}(P,M/M_1)^{\mathcal{O}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi-1} & & \downarrow{\scriptstyle \varphi-1} & & \downarrow{\scriptstyle \varphi-1} & & \\
0 & \longrightarrow & \mathrm{Hom}(P,M_1)^{\mathcal{O}} & \longrightarrow & \mathrm{Hom}(P,M)^{\mathcal{O}} & \longrightarrow & \mathrm{Hom}(P,M/M_1)^{\mathcal{O}} & \longrightarrow & 0
\end{array}$$

By the snake lemma we therefore obtain (functorial) long exact sequences

$$\begin{aligned}
0 \to \mathrm{Hom}_{\mathrm{BK}}(P,M_1) \to \mathrm{Hom}_{\mathrm{BK}}(P,M) \to \mathrm{Hom}_{\mathrm{BK}}(P,M/M_1) \to \\
\to \mathrm{Ext}^1_{\mathrm{SD}}(P,M_1) \to \mathrm{Ext}^1_{\mathrm{SD}}(P,M) \to \mathrm{Ext}^1_{\mathrm{SD}}(P,M/M_1) \to 0
\end{aligned}$$

Likewise we obtain a long exact sequence for any exact sequence in P. One can also produce similar long exact sequences for the groups $\mathrm{Ext}^1_{k_E}, \mathrm{Ext}^1_{\mathrm{eff}}$ and $\mathrm{Ext}^1_{[0,p]}$ but we shall only use the long exact sequence considered above.

CHAPTER 5

Strong Divisibility and Galois Actions

NOTATION 5.0.1. For this section we assume that $K = K_0$. Note that while this assumption does not alter the categories $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ and $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ it does alter how we view $k[[u]]$ as a subring of $\mathcal{O}_{C^\flat}$ (see Notation 2.4.1). In particular the assumption $K = K_0$ implies $v^\flat(u) = 1$.

We have already seen the result of Gee–Liu–Savitt which says that any crystalline representation with Hodge–Tate weights in $[0,p]$ gives rise to an object of $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ via $T \mapsto M(T)/\varpi$. The following theorem is the crucial observation which allows Gee–Liu–Savitt to prove this. Note that unlike Theorem 4.5.1 this result requires no restriction on the Hodge–Tate weights (also a variant holds without the assumption that $K = K_0$).

THEOREM 5.0.2 (Gee–Liu–Savitt). *Let T be a crystalline $\mathcal{O}$-lattice and $M = M(T)/\varpi$. Then there exists a continuous $\mathcal{O}_{C^\flat} \otimes_{\mathbb{F}_p} k_E$-semilinear, φ-equivariant action of G_K on $M \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ which satisfies the following.*

- *For all $m \in M$ and all $\sigma \in G_K$*

$$(\sigma - 1)(m) \in M \otimes_{k[[u]]} u^{\frac{p}{p-1}}\mathcal{O}_{C^\flat}$$

- *For all $m \in M$ and $\sigma \in G_{K_\infty}$*

$$(\sigma - 1)(m) = 0$$

PROOF. When $p > 2$ this is essentially [**16**, Corollary 5.10] (note they consider M with a frobenius twist which is why $p^2/p-1$ appears in *loc. cit.*). We give a proof (which allows $p=2$) in the final chapter. The theorem follows by reducing Lemma 9.2.13 modulo ϖ, using that $v^\flat(u) = 1$ and $v^\flat(\varphi^{-1}(\overline{\mu})) = 1/(p-1)$ (where $\overline{\mu}$ denotes the image of $\mu \in A_{\mathrm{inf}}$ in $\mathcal{O}_{C^\flat}$). □

1. Crystalline Galois Actions

DEFINITION 5.1.1. Let $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$. We say that M is χ_{cyc}^p-free if $\mathrm{Weight}(M) \subset [0,p]$ and M admits a composition series

$$0 = M_n \subset \ldots \subset M_0 = M$$

(see Construction 4.2.4) such that if $M_i/M_{i+1} \cong \overline{\mathfrak{S}}(\{p\};a)$ for some $a \in k_E^\times$ then $M_j/M_{j+1} \not\cong \overline{\mathfrak{S}}(\{0\};b)$ for any $b \in k_E^\times$ and any $j > i$. Pictorially, if we express M as

$$\begin{pmatrix} \ddots & * & * \\ 0 & M_1/M_2 & * \\ 0 & 0 & M_0/M_1 \end{pmatrix}$$

then if a block is isomorphic to $\overline{\mathfrak{S}}(\{p\}; a)$ then no block above it is isomorphic to $\overline{\mathfrak{S}}(\{0\}; b)$.

The aim of this chapter is to prove the following theorem.

THEOREM 5.1.2. *Let $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ be χ_{cyc}^p-free. Then $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ if and only if there exists a continuous $\mathcal{O}_{C^\flat} \otimes_{\mathbb{F}_p} k_E$-semilinear, φ-equivariant action of G_K on $M \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ which satisfies the following.*

- *For all $m \in M$ and all $\sigma \in G_K$*
$$(\sigma - 1)(m) \in M \otimes_{k[[u]]} u^{\frac{p}{p-1}} \mathcal{O}_{C^\flat}$$
- *For all $m \in M$ and $\sigma \in G_{K_\infty}$*
$$(\sigma - 1)(m) = 0$$

DEFINITION 5.1.3. We say an object $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ admits a crystalline[1] G_K-action if $M \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ has a G_K-action as in the theorem.

REMARK 5.1.4. If $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ admits a crystalline G_K-action then we obtain a G_K-action on $T(M) = (M \otimes_{k[[u]]} C^\flat)^{\varphi = 1}$ which extends the G_{K_∞}-action and makes the identification (from Proposition 2.4.6)

$$M \otimes_{k[[u]]} C^\flat = T(M) \otimes_{\mathbb{F}_p} C^\flat$$

G_K-equivariant. The continuity of the G_K-action on $M \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ implies the G_K-representation $T(M)$ is continuous (since the subspace topology on $\mathbb{F}_p \subset C^\flat$ is the discrete topology).

REMARK 5.1.5. Let M and $N \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ and suppose both admit crystalline G_K-actions. In order that a morphism $M \to N$ induces a G_K-equivariant map $M \otimes_{k[[u]]} \mathcal{O}_{C^\flat} \to N \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ it is necessary and sufficient that the induced map $T(M) \to T(N)$ is G_K-equivariant (where the G_K-actions on $T(M)$ and $T(N)$ are as in Remark 5.1.4). Necessity is clear. To see sufficiency observe that, since $M \otimes_{k[[u]]} C^\flat = T(M) \otimes_{\mathbb{F}_p} C^\flat$ is G_K-equivariant and likewise with M replaced by N, if $T(M) \to T(N)$ is G_K-equivariant then so is $M \otimes_{k[[u]]} C^\flat \to N \otimes_{k[[u]]} C^\flat$. Thus $M \otimes_{k[[u]]} \mathcal{O}_{C^\flat} \to N \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ is also.

[1]The term crystalline here comes from Theorem 5.0.2. While we have not yet written out the details, we believe that an easy modification of the arguments in Lemma 9.2.13 allow Theorem 5.0.2 to be proven for semi-stable representations, except that for $m \in M$ we only have

$$(\sigma - 1)(m) \in M \otimes_{k[[u]]} u^{1/p-1} \mathcal{O}_{C^\flat}$$

The loss of a factor of u in this semistable result is due to that fact that the monodromy N in the semistable case is non-zero.

Every rank one object of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ admits a crystalline G_K-action, and this action can be expressed explicitly.

LEMMA 5.1.6. *Let $M = \overline{\mathfrak{S}}(\{r_\tau\}; a)$ (Notation 4.6.2). Then M admits a unique crystalline G_K-action given by*

$$\sigma(e_\tau) = \eta(\sigma)^{\Theta_\tau} e_\tau$$

where $\Theta_\tau = \sum_0^{f-1} r_{\tau\circ\varphi^i} p^i$ and $\eta(\sigma)$ is the unique $p^f - 1$-th root of $\epsilon(\sigma) = \sigma(u)/u \in \mathbb{Z}_p(1) \subset \mathcal{O}_{C^\flat}$ such that $\eta(\sigma) \equiv 1$ modulo $\mathfrak{m}_{C^\flat}$.

PROOF. Theorem 5.0.2 and Proposition 4.6.6 together imply M admits a crystalline G_K-action. Since there is at most one way to extend a G_{K_∞}-valued character to a G_K-valued character (Corollary 3.2.8) Remark 5.1.5 implies this crystalline G_K-action is unique.

In fact the previous paragraph shows there can exist at most one continuous φ-equivariant and k_E-equivariant G_K-action on $M \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ which is trivial on M when restricted to G_{K_∞}. One easily checks that such a G_K-action must be of form given in the lemma (thus to prove the lemma it is not necessary to check that $\eta(\sigma)^{\Theta_\tau} - 1 \in u^{p/p-1}\mathcal{O}_{C^\flat}$; though this can be checked by hand, see Lemma 5.6.8). ☐

2. Induction and Restriction

NOTATION 5.2.1. Let L/K be the unramified extension corresponding to a finite extension l/k, and let $L_\infty = K_\infty L$. Set $\mathfrak{S}_L = W(l)[[u]]$. Extension of scalars along the inclusion $f : \mathfrak{S} \to \mathfrak{S}_L$ describes a functor

$$f^* : \mathrm{Mod}_K^{\mathrm{BK}} \to \mathrm{Mod}_L^{\mathrm{BK}}$$

For $M \in \mathrm{Mod}_K^{\mathrm{BK}}$ the module $f^*M = M \otimes_{\mathfrak{S}} \mathfrak{S}_L$ is made into a Breuil–Kisin module via the semilinear map $m \otimes s \mapsto \varphi_M(m) \otimes \varphi(s)$: this map induces the isomorphism

$$(\varphi^* f^* M)[\tfrac{1}{E}] = (f^* \varphi^* M)[\tfrac{1}{E}] = f^*(\varphi^* M[\tfrac{1}{E}]) \xrightarrow{f^*\varphi_M} f^*(M[\tfrac{1}{E}]) = (f^*M)[\tfrac{1}{E}]$$

with the first $=$ coming from the fact that $\varphi \circ f = f \circ \varphi$. The natural isomorphism

$$f^*M \otimes_{\mathfrak{S}_L} A_{\mathrm{inf}}[\tfrac{1}{\varphi^{-1}(\mu)}] \cong M \otimes_{\mathfrak{S}} A_{\mathrm{inf}}[\tfrac{1}{\varphi^{-1}(\mu)}]$$

is clearly φ, G_{L_∞}-equivariant. It follows from Proposition 2.4.6 that $T(f^*M) = T(M)|_{G_{L_\infty}}$.

NOTATION 5.2.2. With notation as in Notation 5.2.1, restriction of scalars along f induces a functor

$$f_* : \mathrm{Mod}_L^{\mathrm{BK}} \to \mathrm{Mod}_K^{\mathrm{BK}}$$

If $M \in \mathrm{Mod}_L^{\mathrm{BK}}$ we equip f_*M with the obvious semilinear map $m \mapsto \varphi_M(m)$. Let us verify that this makes f_*M into a Breuil–Kisin module. The semilinear map induces the composite:

$$(\varphi^* f_* M)[\tfrac{1}{E}] \to (f_* \varphi^* M)[\tfrac{1}{E}] = f_*(\varphi^* M[\tfrac{1}{E}]) \xrightarrow{f_*\varphi_M} f_*(M[\tfrac{1}{E}]) = (f_*M)[\tfrac{1}{E}]$$

which we claim is an isomorphism. It suffices to check the natural map $\varphi^* f_* M \to f_* \varphi^* M$ is an isomorphism, and this follows because the commutative diagram

$$\begin{array}{ccc} \mathfrak{S} & \xrightarrow{f} & \mathfrak{S}_L \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi} \\ \mathfrak{S} & \xrightarrow{f} & \mathfrak{S}_L \end{array}$$

is a pushout.

LEMMA 5.2.3. *For all* $M \in \mathrm{Mod}_K^{\mathrm{BK}}$ *and* $N \in \mathrm{Mod}_L^{\mathrm{BK}}$ *there are functorial isomorphisms*

$$\mathrm{Hom}(M, f_* N) \cong f_* \mathrm{Hom}(f^* M, N)$$

in $\mathrm{Mod}_K^{\mathrm{BK}}$.

PROOF. The standard adjunction between f^* and f_* provides functorial $\mathfrak{S}$-linear isomorphisms $\mathrm{Hom}_{\mathfrak{S}}(M, f_* N) \to \mathrm{Hom}_{\mathfrak{S}_L}(f^* M, N)$. Explicitly this map sends α onto the homomorphism $m \otimes s \mapsto s\alpha(m)$. As this is φ-equivariant we get isomorphisms as claimed. $\square$

LEMMA 5.2.4. *Let* $N \in \mathrm{Mod}_L^{\mathrm{BK}}$. *Then there are functorial identifications* $\iota_N : T(f_* N) \to \mathrm{Ind}_{L_\infty}^{K_\infty} T(N)$ *such that the diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{BK}}(M, f_* N) & \xrightarrow{\quad 5.2.3 \quad} & \mathrm{Hom}_{\mathrm{BK}}(f^* M, N) \\ \downarrow{\scriptstyle g \mapsto \iota_N \circ T(g)} & & \downarrow{\scriptstyle T} \\ \mathrm{Hom}_{G_{K_\infty}}(T(M), \mathrm{Ind}_{L_\infty}^{K_\infty} T(N)) & \xrightarrow{3.1.1} & \mathrm{Hom}_{G_{L_\infty}}(T(M)|_{G_{L_\infty}}, T(N)) \end{array}$$

commutes for all $M \in \mathrm{Mod}_K^{\mathrm{BK}}$. *The horizontal arrows are obtained from the identifications in Lemma 5.2.3 and Lemma 3.1.1 by respectively taking* φ*-invariants and* G_{K_∞}*-invariants.*

PROOF. Let $\mathcal{O}_{\mathcal{E},L}$ be the p-adic completion of $\mathfrak{S}_L[\frac{1}{u}]$. The map $f : \mathfrak{S} \to \mathfrak{S}_L$ extends to a map $f : \mathcal{O}_{\mathcal{E}} \to \mathcal{O}_{\mathcal{E},L}$ and so we can make sense of the operations f^* and f_* on etale φ-modules. Write $M^{\mathrm{et}} = M \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ and $N^{\mathrm{et}} = N \otimes_{\mathfrak{S}_L} \mathcal{O}_{\mathcal{E},L}$. Then clearly $f^*(M^{\mathrm{et}}) = (f^* M)^{\mathrm{et}}$, and because $\mathcal{O}_{\mathcal{E},L} = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{S}_L$ we also have that $f_*(N^{\mathrm{et}}) = (f_* N)^{\mathrm{et}}$. We obtain maps

$$\mathrm{Hom}_{\mathrm{BK}}(M, f_* N) \to \mathrm{Hom}_{\mathrm{et}}(M^{\mathrm{et}}, f_* N^{\mathrm{et}}), \qquad \mathrm{Hom}_{\mathrm{BK}}(f^* M, N) \to \mathrm{Hom}_{\mathrm{et}}(f^* M^{\mathrm{et}}, N^{\mathrm{et}})$$

which commute with T. The analogue of Lemma 5.2.3 in the setting of etale φ-modules is proved in exactly the same way, and the obtained identification is compatible with maps above. Thus to prove the lemma we may replace $\mathrm{Hom}_{\mathrm{BK}}$ with $\mathrm{Hom}_{\mathrm{et}}$ (homsets in the category of etale φ-modules) and M and N with M^{et} and N^{et} in the diagram of the lemma.

Since $M^{\mathrm{et}} \mapsto T(M^{\mathrm{et}})$ is an equivalence of categories the map $(3.1.1) \circ T \circ (5.2.3) \circ T^{-1}$ describes an identification

$$\mathrm{Hom}_{G_{K_\infty}}(V, T(f_* N)) \to \mathrm{Hom}_{G_{K_\infty}}(V, \mathrm{Ind}_{L_\infty}^{K_\infty} T(N)) \tag{5.2.5}$$

for any continuous G_{K_∞}-representation V on a finitely generated $\mathbb{Z}_p$-module. As (5.2.5) is functorial in V Yoneda's lemma provides the isomorphism ι_N. As (5.2.5) is functorial in N we see that ι_N is functorial. $\square$

LEMMA 5.2.6. *Assume that $k \subset l \subset k_E$.*

(1) *If $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ then $f^*M \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$ and for each $\theta : l \to k_E$ we have*
$$\mathrm{Weight}_\theta(f^*M) = \mathrm{Weight}_{\theta|_k}(M)$$
(2) *If $N \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$ then $f_*N \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ and*
$$\mathrm{Weight}_\tau(f_*N) = \bigcup_{\theta|_k=\tau} \mathrm{Weight}_\theta(N)$$

PROOF. By functoriality both f^* and f_* preserve $\mathcal{O}$-actions. Recall from Remark 2.5.2 that for each $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, k_E)$ there is an idempotent $i_\tau \in k[[u]] \otimes_{\mathbb{F}_p} k_E$. Likewise for each $\theta \in \mathrm{Hom}_{\mathbb{F}_p}(l, k_E)$ there is an idempotent $i_\theta \in l[[u]] \otimes_{\mathbb{F}_p} k_E$. Note that the inclusion $k[[u]] \otimes_{\mathbb{F}_p} k_E \to l[[u]] \otimes_{\mathbb{F}_p} k_E$ sends $i_\tau \mapsto \sum_{\theta|_k=\tau} i_\theta$. Thus $(f^*M)_\theta = M_{\theta|_k}$ and $(f_*N)_\tau = \prod_{\theta|_k=\tau} N_\theta$. Both (1) and (2) then follow by verifying the second condition of Lemma 4.4.4. $\square$

LEMMA 5.2.7. *Let $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ and let f be as in Notation 5.2.1. Then $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ if and only if $f^*M \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$.*

PROOF. After Lemma 5.2.6 we only have to show that f^*M being strongly divisible implies M is strongly divisible. Observe that there is an injective morphism $M \to f_*f^*M$ which sends $m \mapsto m \otimes 1$. Since $f_*f^*M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ by Lemma 5.2.6 and $M \to f_*f^*M$ has torsionfree cokernel, Proposition 4.4.5 implies $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$. $\square$

3. The Key Lemma

If $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ we write $M^\flat = M \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$. The purpose of this section is to give the main technical input into the proof of Theorem 5.1.2.

HYPOTHESIS 5.3.1. Consider pairs $P, M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ with the property that P and M admit composition series
$$0 = P_n \subset \ldots \subset P_0 = P, \qquad 0 = M_m \subset \ldots \subset M_0 = M$$
such that for each i and j
$$\mathrm{Hom}(P_i/P_{i+1}, M_j/M_{j+1})^{\mathcal{O}} \not\cong \overline{\mathfrak{S}}(\{-p\}; a)$$
for any $a \in k_E^\times$.

REMARK 5.3.2. Note that if $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ with $\mathrm{Weight}(M) \subset [0, p]$ then M is χ_{cyc}^p-free if there exists a composition series $0 = M_n \subset \ldots \subset M_0 = M$ such that for each i the pair
$$M_i/M_{i+1}, M_{i+1}$$
is as in Hypothesis 5.3.1.

LEMMA 5.3.3. *Let P and M be a pair as in Hypothesis 5.3.1. For any unramified extension L/K, with notation as in Notation 5.2.1, the pair f^*P and f^*M are also as in Hypothesis 5.3.1.*

PROOF. Let (P_i) and (M_j) be composition series of P and M as in Hypothesis 5.3.1. Choose a refinement of the filtration $(f^*P_i)_i$ of f^*P to a composition series, and do the same for M. If these composition series are not as in Hypothesis 5.3.1 then for some i and j there will exist subquotients $\mathcal{P}_i \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$ of $f^*(P_i/P_{i+1})$ and $\mathcal{M}_j \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$ of $f^*(M_j/M_{j+1})$ such that

$$\mathrm{Hom}(\mathcal{P}_i, \mathcal{M}_j)^{\mathcal{O}} \cong \overline{\mathfrak{S}}(\{-p\}; a)$$

For this to be true we must have $\mathcal{P}_i \cong \overline{\mathfrak{S}}(\{p\}; b)$ and $\mathcal{M}_j \cong \overline{\mathfrak{S}}(\{0\}; c)$ with $a = cb^{-1}$. If $\mathcal{P}_i \cong \overline{\mathfrak{S}}(\{p\}; b)$ then $T(\mathcal{P}_i) = \psi_b \chi_{\mathrm{cyc}}^{-1}$ (Lemma 4.6.4) is a subquotient of $T(P_i/P_{i+1})|_{G_{L_\infty}}$. Since $T(P_i/P_{i+1})$ is irreducible it has the form $\mathrm{Ind}_{F_\infty}^{K_\infty} \vartheta$ for some unramified extension F/K and some character ϑ, and so we must have that ϑ equals $\psi_b \chi_{\mathrm{cyc}}^{-1}$ when restricted to $L_\infty F_\infty$. However this implies $T(P_i/P_{i+1})$ is not irreducible unless $F_\infty = K_\infty$. Thus we must have that $P_i/P_{i+1} = \overline{\mathfrak{S}}(\{p\}; b^{1/p^{[l:k]}})$. Likewise if $\mathcal{M}_j = \overline{\mathfrak{S}}(\{0\}; c)$ then $M_j/M_{j+1} = \overline{\mathfrak{S}}(\{0\}; c^{1/p^{[l:k]}})$.[2] But this contradicts the assumption that (P_i) and (M_j) are composition series as in Hypothesis 5.3.1. □

The relevance of the hypothesis is that it allows us to prove the following lemma and corollary.

LEMMA 5.3.4. *Let P and M be as in Hypothesis 5.3.1. Then there exists no sequence $x_i \in \mathrm{Hom}(P, M)^{\mathcal{O},\flat} = \mathrm{Hom}(P, M)^{\mathcal{O}} \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ such that*

$$\varphi(x_i) u^p = x_{i+1} \notin u\,\mathrm{Hom}(P, M)^{\mathcal{O},\flat} \tag{5.3.5}$$

for all $i \geq 0$.

PROOF. Let L/K be a sufficiently large unramified extension such that, with notation as in Notation 5.2.1, composition series of both f^*P and f^*M have irreducible subquotients of rank one. Note that $\mathrm{Hom}(P, M)^{\mathcal{O},\flat} = \mathrm{Hom}_{\mathcal{O}_{C^\flat} \otimes_{\mathbb{F}_p} k_E}(P^\flat, M^\flat)$. Since $(f^*P)^\flat = P^\flat$, and likewise for M, it follows that $\mathrm{Hom}(P, M)^{\mathcal{O},\flat} = \mathrm{Hom}(f^*P, f^*M)^{\mathcal{O},\flat}$. Thus we may replace P and M with f^*P and f^*M (note that f^*P and f^*M satisfy Hypothesis 5.3.1 by Lemma 5.3.3) and therefore we may assume that every irreducible subquotient of P or M is of rank one.

We shall argue by induction on the length of $\mathrm{Hom}(P, M)^{\mathcal{O}}$. If it has length one then both P and M are of rank one and $\mathrm{Hom}(P, M)^{\mathcal{O}} \cong \overline{\mathfrak{S}}(\{r_\tau\}; a)$

[2] Here we have used that if l/k is a finite extension then with notation as in Notation 5.2.1, $f^*\overline{\mathfrak{S}}(\{r_\tau\}; x) = \overline{\mathfrak{S}}(\{s_\theta\}; y)$ where $s_\theta = r_{\theta|_k}$ and $y = x^{1/p^{[l:k]}}$. To check this note that $f^*\overline{\mathfrak{S}}(\{r_\tau\}; x) = \overline{\mathfrak{S}}(\{s_\theta\}; y)$ for some s_θ and $y \in k_E^\times$. That $s_\theta = r_{\theta|_k}$ follows from Lemma 5.2.6(1). That $y = x^{1/p^{[l:k]}}$ follows because the unramified character ψ_y on G_{K_∞} sending the geometric Frobenius to y sends the geometric Frobenius in G_{L_∞} onto $y^{p^{[l:k]}}$.

with $r_\tau \in [-p,p]$ and $a \in k_E^\times$. As P and M are as in Hypothesis 5.3.1 not all the r_τ equal $-p$. For $n \geq 0$ we have

$$u^{\sum_1^n p^{n-i+1}}\varphi^n(x_0) = x_n \notin u\operatorname{Hom}(P,M)^{\mathcal{O},\flat}$$

On the other hand for each τ we can choose a generator e_τ of $\operatorname{Hom}(P,M)^{\mathcal{O}}_\tau$ such that $\varphi(e_{\tau\circ\varphi}) = (a)_\tau u^{r_\tau} e_\tau$. One inductively computes that $u^{\sum_1^f p^{f-i+1}}\varphi^f(e_\tau) = au^{\sum_1^f p^{f-i}(r_{\tau\circ\varphi^i}+p)}e_\tau$ which, since each $r_{\tau\circ\varphi^i}+p \geq 0$ with at least one > 0, is killed by reduction modulo u. As the e_τ generate $\overline{\mathfrak{S}}(\{r_\tau\};a) \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ we get a contradiction.

For the inductive step observe that (with P_1 as in Hypothesis 5.3.1) the pairs P_1, M and $P/P_1, M$ are both as in Hypothesis 5.3.1. Thus, provided P is not of rank one so that $P/P_1, P_1 \neq 0$, we may assume that the lemma holds for the outer terms of the exact sequence

$$0 \to \operatorname{Hom}(P/P_1,M)^{\mathcal{O},\flat} \to \operatorname{Hom}(P,M)^{\mathcal{O},\flat} \to \operatorname{Hom}(P_1,M)^{\mathcal{O},\flat} \to 0$$

Respectively write H_1, H and H_2 for the modules in this sequence (so that $0 \to H_1 \to H \to H_2 \to 0$ is exact). Let $\overline{x}_i$ denote the image of x_i in H_2. If $\overline{x}_0$ is zero then $x_0 \in H_1$ and $x_i \in H_1[\frac{1}{u}] \cap H = H_1$ and not in $H_1[\frac{1}{u}] \cap uH = uH_1$; by our inductive assumption no such x_0 can exist. Thus $\overline{x}_0 \neq 0$ and since $\overline{x}_i$ satisfies (5.3.5) our inductive assumption implies we must have $\overline{x}_i \in uH_2$ for some i. Since all the terms in the above exact sequence are free the sequence stays exact after reducing modulo u. It follows there exists $y_i \in H_1$ such that $y_i \in x_i + uH$. Since $\operatorname{Weight}(\operatorname{Hom}(P,M)^{\mathcal{O}}) \subset [-p,p]$ (see the proof of Proposition 4.7.16) we have $\varphi(H) \subset u^{-p}H$. Thus the inductively defined sequence $y_{i+j+1} = u^p\varphi(y_{i+j})$ is such that y_{i+j} is contained in

$$x_{i+j} + u^{p^j}H \subset x_{i+j} + uH$$

Note that each $y_{i+j} \in H_1$ and further is not contained in uH_1 (if it were then $x_{i+j} \in uH$, a contradiction). Thus $(y_{i+j})_{j\geq 0}$ is a sequence in H_1 as in the lemma. However our inductive assumption implies no such element exists so we obtain a contradiction. Thus, provided P is not of rank one, no such x_0 can exist.

If P has rank one one argues similarly using the exact sequence $0 \to \operatorname{Hom}(P,M_1)^{\mathcal{O}} \to \operatorname{Hom}(P,M)^{\mathcal{O}} \to \operatorname{Hom}(P,M/M_1)^{\mathcal{O}} \to 0$. ☐

Corollary 5.3.6. *Let P and M be as in Hypothesis 5.3.1. Then for any $x \in u^{p/p-1}\operatorname{Hom}(P,M)^{\mathcal{O},\flat}$ the sum*

$$X = -\sum_{n\geq 0}\varphi^n(x)$$

converges in $u^{p/p-1}\operatorname{Hom}(P,M)^{\mathcal{O},\flat}$ and is the unique solution in $u^{p/p-1}\operatorname{Hom}(P,M)^{\mathcal{O},\flat}$ to $(\varphi-1)(X) = x$.

Proof. Let $H = \operatorname{Hom}(P,M)^{\mathcal{O},\flat}$. As $\varphi(H) \subset u^{-p}H$ we see that $x \in u^{\epsilon+p/p-1}H$ implies $\varphi(x) \in u^{p\epsilon+p/p-1}H$ for any $\epsilon \geq 0$. Thus, as $u^{p/p-1}H \subset H$

is closed, if X converges it will do so in $u^{p/p-1}H$. It also follows that if there is an N such that $\varphi^N(x) \in u^{1+p/p-1}H$ then the sum X converges. Suppose the sum does not converge, then $x_n := u^{-p/p-1}\varphi^n(x) \in H \setminus uH$ for all $n \geq 0$. However, as $x_{i+1} = u^p\varphi(x_i)$, the existence of such x_i contradicts Lemma 5.3.4. We conclude that the sum X must converge in $u^{p/p-1}H$. Clearly $(\varphi - 1)(X) = x$. If $(\varphi - 1)(X') = x$ also then $(\varphi - 1)(X - X') = 0$. We can write $X - X' = u^{p/p-1}X_0$ with $X_0 \in H$. Then $u^p\varphi(X_0) = X_0$ and so Lemma 5.3.4 implies $X_0 \in uH$. But then $\varphi^n(u^{p/p-1}X_0)$ converges to zero, and so $X - X' = 0$. $\square$

4. Constructing Galois Actions

In this section we shall prove the only if direction of Theorem 5.1.2. In fact we prove slightly more:

PROPOSITION 5.4.1. *Let $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ and let $0 = M_n \subset \ldots \subset M_0 = M$ be a composition series as in Definition 5.1.1. Then M admits a crystalline G_K-action which is unique amongst those making each $M_i^\flat \subset M^\flat$ G_K-stable.*

We shall argue by induction on the length of M; thus the main part of the proof involves showing that crystalline G_K-actions can be extended along strongly divisible extensions (this will be where we use the results of the previous section).

CONSTRUCTION 5.4.2. If $M, P \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ both admit crystalline G_K-actions then $\mathrm{Hom}(P, M)^{\mathcal{O}}$ admits a crystalline G_K-action given as follows. Let $R^\flat = \mathcal{O}_{C^\flat} \otimes_{\mathbb{F}_p} k_E$. If $f \in \mathrm{Hom}(P, M)^{\mathcal{O},\flat} = \mathrm{Hom}_{R^\flat}(P^\flat, M^\flat)$ then $\sigma(f) = \sigma \circ f \circ \sigma^{-1}$.

CONSTRUCTION 5.4.3. Let $0 \to M \to N \to P \to 0$ be an exact sequence in $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ and suppose M and P admit crystalline G_K-actions. After Construction 4.7.1 we may assume there is an $f \in \mathrm{Hom}(P, M)^{\mathcal{O}}[\frac{1}{u}]$ such that $N = M \oplus P$ with frobenius given by

$$\varphi_N(m, n) = (\varphi_M(m) + f \circ \varphi_P(z), \varphi_P(z))$$

To give N a crystalline G_K-action making $0 \to M^\flat \to N^\flat \to P^\flat \to 0$ equivariant is to give $f_\sigma \in \mathrm{Hom}(P, M)^{\mathcal{O},\flat} = \mathrm{Hom}_{R^\flat}(P^\flat, M^\flat)$ for each $\sigma \in G_K$ such that

$$\sigma(m, z) = (\sigma(m) + f_\sigma \circ \sigma(z), \sigma(z))$$

satisfying the following conditions:

- The fact that the action of σ on N is a continuous group action translates to asking that $f_\sigma \in Z^1(G_K, \mathrm{Hom}(P, M)^{\mathcal{O},\flat})$ (continuous 1-cocycles), i.e. that

$$f_{\sigma\tau} = f_\sigma + \sigma(f_\tau)$$

where the action of G_K on f_σ is as in Construction 5.4.2.

- The fact that the G_K-action is φ-equivariant translates to
$$(\varphi - 1)(f_\sigma) = (\sigma - 1)(f)$$
- The fact that $(\sigma - 1)(n) \in u^{p/p-1}N^\flat$ for $n \in N$ translates to
$$f_\sigma \circ \sigma(z) \in u^{p/p-1}M^\flat$$
for each $z \in P$. Since $(\sigma - 1)(z) \in u^{p/p-1}P^\flat$ this is equivalent to asking that $f_\sigma \in u^{p/p-1}\operatorname{Hom}(P, M)^{\mathcal{O},\flat}$.
- Since $(\sigma - 1)(n) = 0$ for $\sigma \in G_{K_\infty}$ and $n \in N$, and the same is true for $m \in M$ and $z \in P$, it follows that $f_\sigma(z) = 0$ for $z \in P$ and $\sigma \in G_{K_\infty}$. Thus $f_\sigma = 0$ whenever $\sigma \in G_{K_\infty}$.

Lemma 5.4.4. *Let P and M be as in Hypothesis 5.3.1 and assume further that both P and M admit crystalline G_K-actions. If $0 \to M \to N \to P \to 0$ is an exact sequence in $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ then N admits a unique crystalline G_K-action making $0 \to M^\flat \to N^\flat \to P^\flat \to 0$ equivariant for the G_K-action.*

When P and M are both rank one the uniqueness of such an action was observed in [**16**, Lemma 8.1].

Proof. Keep the notation of Construction 5.4.3. Since N is assumed to be strongly divisible we may choose f so that $f \in \operatorname{Hom}(P, M)^{\mathcal{O}}$ (Remark 4.7.10). As the G_K-action on $\operatorname{Hom}(P, M)^{\mathcal{O}}$ is crystalline $(\sigma - 1)(f) \in u^{p/p-1}\operatorname{Hom}(P, M)^{\mathcal{O},\flat}$ and so Lemma 5.3.6 tells us that
$$f_\sigma = -\sum_{n=0}^{\infty} \varphi^n((\sigma - 1)(f))$$
is the unique element in $u^{p/p-1}\operatorname{Hom}(P, M)^{\mathcal{O},\flat}$ satisfying $(\varphi - 1)(f_\sigma) = (\sigma - 1)(f)$. For $\sigma \in G_{K_\infty}$ we have $(\sigma - 1)(f) = 0$ so $f_\sigma = 0$. Since $(\sigma - 1)(f)$ is a continuous 1-cocycle, and φ and σ commute, the same is true for f_σ. Thus f_σ satisfies all the necessary conditions to induce a crystalline G_K-action on N. This proves the lemma. $\square$

Proof of Proposition 5.4.1. First suppose that each of the subquotients M_i/M_{i+1} are of rank one. If M is of rank one itself then we have already seen that M admits a unique crystalline G_K-action. For general M we have an exact sequence
$$0 \to M_1 \to M \to M/M_1 \to 0$$
Arguing inductively on the length of M we may assume M_1 and M/M_1 admit unique crystalline G_K-actions such that each $M_i^\flat \subset M_1^\flat$ is G_K-stable. Since M is strongly divisible this is an exact sequence in $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$. The pair M/M_1 and M_1 are as in Hypothesis 5.3.1 and so Lemma 5.4.4 implies M admits a unique crystalline G_K-action extending the G_K-action on $M_1^\flat \subset M^\flat$. This proves the proposition when each of the M_i/M_{i+1} are rank one.

Now we prove the proposition when M is irreducible (i.e. $T(M)$ is irreducible as a G_{K_∞}-representation) but not of rank one. Let L/K be the unramified extension of degree equal to the k_E-dimension of $T(M)$ so that $T(M) = \mathrm{Ind}_{L_\infty}^{K_\infty} \chi$ for some character χ and $T(f^*M) = \bigoplus_\gamma \chi^{(\gamma)}$. It follows that any composition series of f^*M has rank one subquotients and is a composition series as in Definition 5.1.1 (if not then f^*M must have $\overline{\mathfrak{S}}(\{p\}; a)$ as a subquotient and so $\chi^{(\gamma)} = \psi_a \chi_{\mathrm{cyc}}^{-1}$ for some γ. This contradicts the irreducibility of $T(M)$). The previous paragraph then implies f^*M admits a crystalline G_L-action which induces a G_L-action on $T(f^*M)$ preserving the direct sum decomposition; in particular this G_L-action induces a continuous extension of χ from G_{L_∞} to G_L.

It follows that the G_L-action on $T(f^*M) = T(M)$ extends to a unique continuous G_K-action because we can identify $T(M) = \mathrm{Ind}_L^K \chi$ (see the proof of Lemma 3.2.5). Note this action is compatible with the G_{K_∞}-action on $T(M)$. Via the identification

$$M \otimes_{k[[u]]} C^\flat = T(M) \otimes_{\mathbb{F}_p} C^\flat$$

we obtain a continuous G_K-action on $M^\flat \otimes C^\flat$. We claim this gives a crystalline G_K-action on M, i.e. we claim that (1) $\sigma(m) = m$ for all $m \in M$ and $\sigma \in G_{K_\infty}$, and (2) that $(\sigma - 1)(m) \in u^{p/p-1} M^\flat$ for all $m \in M$ and $\sigma \in G_K$ (so that in particular $M^\flat$ is G_K-stable). Observe that (1) holds because by construction the G_K-action on $T(M)$ extends the G_{K_∞}-action. Since the G_L-action is a crystalline G_L-action (2) holds for $\sigma \in G_L$. Using the lemma below we conclude that (2) holds for all $\sigma \in G_K$, which proves the proposition when M is irreducible.

LEMMA 5.4.5. *Every $\sigma \in G_K$ can be written (non-uniquely) as a product $\sigma_{\mathrm{ur}}\sigma_\infty$ with $\sigma_{\mathrm{ur}} \in G_L$ and $\sigma_\infty \in G_{K_\infty}$.*

PROOF. Since K_∞ is totally wildly ramified the natural inclusion induces an isomorphism $G_{K_\infty}/G_{L_\infty} \to G_K/G_L$ where $L_\infty = LK_\infty$ (see the proof of Proposition 3.2.4) and so we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_L & \longrightarrow & G_K & \longrightarrow & \mathrm{Gal}(L/K) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & G_{L_\infty} & \longrightarrow & G_{K_\infty} & \longrightarrow & \mathrm{Gal}(L_\infty/K_\infty) & \longrightarrow & 1
\end{array}$$

It is easy to deduce the lemma from this diagram. $\square$

Note this G_K-action on $M^\flat$ is unique because there is only one way to extend the G_{K_∞}-action on $T(M)$, by Corollary 3.2.8.

To finish the proof of the proposition we again induct on the length of M. The previous paragraph establishes the base case. The inductive step proceeds exactly as it did in the first paragraph. $\square$

5. Functoriality and Uniqueness

In this section we briefly discuss the extent with which the construction of the previous section, which associates to any χ_{cyc}^p-free object of $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$

a crystalline G_K-action, is functorial or unique. For the following reasons we are not able to say anything in general.

- Our construction of the crystalline G_K-actions in Proposition 5.4.1 requires a choice of composition series. It is unclear to us whether the obtained G_K-action depends upon this choice. Thus it is unclear whether this G_K-action is unique.
- We also do not know whether, if $M \to N$ is a morphism of χ_{cyc}^p-free objects in $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ and (M_i) is a composition series of M as in Definition 5.1.1, the image of this filtration in N can be extended to a composition series of N again satisfying the conditions of Definition 5.1.1. Being unable to do this we are not able to deduce any kind of functoriality statements.

On the other hand if we assume the following stronger cyclotomic-freeness result (in the spirit of Notation 3.3.1) then we are able to say something about functoriality and uniqueness.

NOTATION 5.5.1. Let V be a continuous representation of G_{K_∞} on a finite dimensional k_E-vector space. We say that V is cyclotomic-free it admits a composition series $0 = V_n \subset \ldots \subset V_0 = V$ such that for each i

$$H^0(G_{K_\infty}, \mathrm{Hom}(V_i/V_{i+1} \otimes \mathbb{Z}_p(1), V_j/V_{j+1})) = 0$$

for $j > i$. Note that if V is a G_K-representation then V is cyclotomic-free in the sense of Notation 3.3.1 if and only if $V|_{G_{K_\infty}}$ is cyclotomic-free (using the results of Chapter 3).

REMARK 5.5.2. If $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ is such that $T(M)|_{G_{L_\infty}}$ is cyclotomic-free for some unramified extension L/K then M is χ_{cyc}^p-free. Note that it is not however sufficient only that $T(M)$ be cyclotomic-free. For instance if

$$M = k[[u]]^2, \qquad \varphi = \begin{pmatrix} 1 & 1 \\ 0 & au^p \end{pmatrix}$$

with $a \in k_E^\times$ not equal to 1 then M is not χ_{cyc}^p-free but $T(M)$ is cyclotomic-free.

PROPOSITION 5.5.3. *Let $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ be χ_{cyc}^p-free and be such that $T(M)$ is cyclotomic-free. Then M admits a unique crystalline G_K-action and if $p > 2$ the association $M \mapsto T(M)$ from $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ to G_K-representations is functorial.*

PROOF. If M admits two different crystalline G_K-actions then by Remark 5.1.5 we obtain two different extensions of the G_{K_∞}-action on $T(M)$ to a G_K-action. By Theorem 3.3.9 this is impossible, and so the crystalline G_K-action on M is unique. For functoriality; if $M \to N$ is a morphism of Breuil–Kisin modules as in the proposition, then we obtain a map of G_{K_∞}-representations $T(M) \to T(N)$. If $p > 2$, Theorem 3.3.10 implies this map must be G_K-equivariant for the G_K-actions on $T(M)$ and $T(N)$. $\square$

EXAMPLE 5.5.4. We conclude this section by giving an example of an $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ which admits two different crystalline G_K-actions. We point out that this example is not χ_{cys}^p-free. Take $k = \mathbb{F}_p$ and let M be the strongly divisible Breuil–Kisin module

$$M = k_E[[u]]^2, \qquad \varphi_M = \begin{pmatrix} 1 & 0 \\ 0 & u^p \end{pmatrix}$$

The associated G_{K_∞}-representation fits into a split exact sequence $0 \to \mathbb{F}_p \to T(M) \to \mathbb{F}_p(\chi_{\mathrm{cyc}}) \to 0$. If $c_\sigma \in Z^1(G_K, \mathbb{F}_p(\chi_{\mathrm{cyc}}))$ is a 1-cocycle such that $c_\sigma|_{G_{K_\infty}} = 0$ then we can equip M with the crystalline G_K-action

$$\sigma = \begin{pmatrix} 1 & \sigma(u^{p/p-1})c_\sigma \\ 0 & \eta(\sigma)^p \end{pmatrix}$$

Here $u^{1/p-1}$ is a fixed choice of $p-1$-th root of u in $\mathcal{O}_{C^\flat}$. Note that if e_1 and e_2 denotes the standard basis of M then $T(M)$ is generated over $\mathbb{F}_p$ by e_1 and $u^{-p/p-1}e_2$. Since

$$\sigma(e_1, u^{-p/p-1}e_2) = (e_1, u^{-p/p-1}e_2) \begin{pmatrix} 1 & c_\sigma \\ 0 & \eta(\sigma)^p\sigma(u^{p/p-1}u^{-p/p-1}) \end{pmatrix} = (e_1, u^{-p/p-1}e_2) \begin{pmatrix} 1 & c_\sigma \\ 0 & \chi_{\mathrm{cyc}}^{-1}(\sigma) \end{pmatrix}$$

describes a continuous G_K-action on $T(M)$ it follows that σ defines a continuous φ-equivariant G_K-action on $M \otimes_{k[[u]]} \mathcal{O}_C^\flat$. It is a crystalline G_K-action because

$$\sigma - 1 = \begin{pmatrix} 0 & \sigma(u^{p/p-1})c_\sigma \\ 0 & \eta(\sigma)^p - 1 \end{pmatrix} \in u^{p/p-1}\,\mathrm{M}_2(\mathcal{O}_{C^\flat})$$

If we choose $c_\sigma = 0$ then $T(M)$ with the induced G_K-action fits into a split exact sequence $0 \to \mathbb{F}_p \to T(M) \to \mathbb{F}_p(\chi_{\mathrm{cyc}}^{-1}) \to 0$. However, if we choose an identification $\mu_p(\overline{K}) = \mathbb{F}_p$ and set $c_\sigma = \sigma(\pi^{1/p})/\pi^{1/p}$ then $c_\sigma|_{G_{K_\infty}} = 0$ and c_σ represents a non-trivial class in $H^1(G_K, \mathbb{F}_p(\chi_{\mathrm{cyc}}))$. Therefore $T(M)$ with the induced G_K-action fits into a non-split exact sequence $0 \to \mathbb{F}_p \to T(M) \to \mathbb{F}_p(\chi_{\mathrm{cyc}}^{-1}) \to 0$. This shows that a strongly divisible Breuil–Kisin module may admit multiple crystalline G_K-actions.

6. Completing the Proof of Theorem 5.1.2

The aim of this section is to finish the proof of Theorem 5.1.2. First we will need the following lemma.

LEMMA 5.6.1. *Let $N \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ admit a crystalline G_K-action, and let $0 \to T_1 \to T(N) \to T_2 \to 0$ be an exact sequence of G_K-representations (with the G_K-action on $T(N)$ as in Remark 5.1.4). Let*

$$0 \to M \to N \to P \to 0$$

be the corresponding exact sequence in $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ (Lemma 4.2.3). Then the G_K-action on $N \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ induces a crystalline G_K-action on M and P.

PROOF. By hypothesis both T_1 and T_2 have G_K-actions making the exact sequence

$$0 \to T_1 \otimes_{\mathbb{F}_p} C^\flat \to T(N) \otimes_{\mathbb{F}_p} C^\flat \to T_2 \otimes_{\mathbb{F}_p} C^\flat \to 0$$

G_K-equivariant. Since $M \otimes_{k[[u]]} C^\flat = T_1 \otimes_{\mathbb{F}_p} C^\flat$ it follows that if $m \in M$ and $\sigma \in G_K$ then $\sigma(m) \in M \otimes_{k[[u]]} C^\flat$. Thus

$$\sigma(m) - m \in (N \otimes_{k[[u]]} u^{p/p-1}\mathcal{O}_{C^\flat}) \cap (M \otimes_{k[[u]]} C^\flat)$$

Since the quotient $N/M = P$ is free over $k[[u]]$ this intersection is equal to $M \otimes_{k[[u]]} u^{p/p-1}\mathcal{O}_{C^\flat}$. Thus the crystalline G_K-action on N restricts to a crystalline G_K-action on M. This implies that the crystalline G_K-action on $N \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ descends through the surjection $N \otimes_{k[[u]]} \mathcal{O}_{C^\flat} \to P \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$ to a G_K-action satisfying $\sigma(z) - z \in P \otimes_{k[[u]]} u^{p/(p-1)}\mathcal{O}_{C^\flat}$ for all $z \in P$. $\square$

PROOF OF THEOREM 5.1.2. We need to show that if M admits a crystalline G_K-action and has $\mathrm{Weight}(M) \subset [0,p]$ then $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$. After Lemma 5.2.7 it suffices to show f^*M is strongly divisible for some unramified extension L/K. Arguing as in the proof of Lemma 5.3.3 we see that since M is χ_{cyc}^p-free, so is f^*M. Thus we may suppose that each irreducible subquotient of M is of rank one.

We shall argue by induction on the length of M. If M is of rank one itself then there is nothing to prove since all rank one objects of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ with weights contained in $[0,p]$ are strongly divisible. If M has length greater than one then we fit M into an exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0$$

where M_1 is as in Hypothesis 5.1.1. Thus M/M_1 is of rank one. Since $\mathrm{Weight}(M) \subset [0,p]$ Lemma 4.7.8 implies M represents a class in $\mathrm{Ext}^1_{[0,p]}(M/M_1, M_1)$. By Lemma 5.6.1 both M/M_1 and M_1 admit crystalline G_K-actions and so by induction we may suppose both M/M_1 and M_1 are strongly divisible. Thus, to finish the proof we have to show that any extension in $\mathrm{Ext}^1_{[0,p]}(M/M_1, M_1)$ admitting a crystalline G_K-action as in Construction 5.4.3 lies in $\mathrm{Ext}^1_{\mathrm{SD}}(M/M_1, M_1)$. Since the pair M/M_1 and M_1 are as in Hypothesis 5.3.1 this follows from Proposition 5.6.3 below. $\square$

CONSTRUCTION 5.6.2. For $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ satisfying $\mathrm{Weight}(M) \subset [-p,\infty]$ and which are endowed with a crystalline G_K-action, define

$$M^{\mathrm{Gal}} \subset u^{-p}M$$

to be the set of f such that there exists a continuous 1-cocycle $f_\sigma \in Z^1(G_K, M^\flat)$ satisfying the following:

- $(\varphi - 1)(f_\sigma) = (\sigma - 1)(f)$,
- $f_\sigma \in u^{p/p-1}M^\flat$, and
- $f_\sigma = 0$ for $\sigma \in G_{K_\infty}$.

These are precisely the conditions required in Construction 5.4.3. Note that if $m \in M$ then $(\varphi - 1)(m) \in M^{\mathrm{Gal}}$: since $\mathrm{Weight}(M) \subset [-p, \infty]$ we have $\varphi(M) \subset u^{-p}M$ so $(\varphi - 1)(m) \in u^{-p}M$ and we can take $f_\sigma = (\sigma - 1)(m)$. Thus it makes sense to define $H^1_{\mathrm{Gal}}(M)$ to be the cokernel of

$$\varphi - 1 : M \to M^{\mathrm{Gal}}$$

Note that the inclusion of $M^{\mathrm{Gal}} \to M[\frac{1}{u}]$ induces an inclusion $H^1_{\mathrm{Gal}}(M) \to H^1(M)$. Note also that if $M \to N$ is a morphism preserving crystalline G_K-actions then M^{Gal} is mapped into N^{Gal}. Thus $H^1_{\mathrm{Gal}}(M)$ is functorial in M (for morphisms respecting crystalline G_K-actions).

If P and $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ both admit crystalline G_K-actions and have weights in $[0,p]$ then $\mathrm{Hom}(P,M)^{\mathcal{O}}$ has weights in $[-p,p]$ and so it makes sense to define

$$\mathrm{Ext}^1_{\mathrm{Gal}}(P,M) \subset \mathrm{Ext}^1_{[0,p]}(P,M) \subset \mathrm{Ext}^1_{k_E}(P,M)$$

as the submodule identified with $H^1_{\mathrm{Gal}}(\mathrm{Hom}(P,M)^{\mathcal{O}})$ (where we give $\mathrm{Hom}(P,M)^{\mathcal{O}}$ the crystalline G_K-action as in Construction 5.4.2) under (4.7.3). Again this is functorial in both P and M (for morphisms respecting crystalline G_K-actions).

Proposition 5.6.3. *Let P and M be as in Hypothesis 5.3.1 and assume both admit crystalline G_K-actions. Assume that P is of rank one and all irreducible subquotients of M are of rank one. Then* $\mathrm{Ext}^1_{\mathrm{Gal}}(P,M) = \mathrm{Ext}^1_{\mathrm{SD}}(P,M)$.

Using the following lemma we shall reduce this proposition to the case when M is of rank one.

Lemma 5.6.4. *Let P and M be as in Hypothesis 5.3.1 and further assume that both P and M admit crystalline G_K-actions. Let $0 \to M_1 \to M \to M/M_1 \to 0$ be an exact sequence in $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ obtained as in Lemma 5.6.1 and assume that P and M_1 are as in Hypothesis 5.3.1. Then there is a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathrm{BK}}(P,M_1) \to \mathrm{Hom}_{\mathrm{BK}}(P,M) \to \mathrm{Hom}_{\mathrm{BK}}(P,M/M_1) \to \\ \to \mathrm{Ext}^1_{\mathrm{Gal}}(P,M_1) \to \mathrm{Ext}^1_{\mathrm{Gal}}(P,M) \to \mathrm{Ext}^1_{\mathrm{Gal}}(P,M/M_1) \end{aligned}$$

Proof. Using the snake lemma it suffices to show that the bottom row of the following diagram is exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(P,M_1)^{\mathcal{O}} & \longrightarrow & \mathrm{Hom}(P,M)^{\mathcal{O}} & \longrightarrow & \mathrm{Hom}(P,M/M_1)^{\mathcal{O}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi-1} & & \downarrow{\scriptstyle \varphi-1} & & \downarrow{\scriptstyle \varphi-1} & & \\ 0 & \longrightarrow & \mathrm{Hom}(P,M_1)^{\mathcal{O},\mathrm{Gal}} & \longrightarrow & \mathrm{Hom}(P,M)^{\mathcal{O},\mathrm{Gal}} & \longrightarrow & \mathrm{Hom}(P,M/M_1)^{\mathcal{O},\mathrm{Gal}} & & \end{array}$$

Let H_1, H and H_2 respectively denote the terms of the top row, so that $0 \to H_1 \to H \to H_2 \to 0$ is exact. We first observe that this extension lies in $\mathrm{Ext}^1_{\mathrm{SD}}(H_2,H_1)$ (note that H_2 and H_1 are not strongly divisible because their weights are not contained in $[0,p]$ but that $\mathrm{Ext}^1_{\mathrm{SD}}(H_2,H_1)$ still makes sense, see the construction given in Variant 4.7.9) because it is obtained from $0 \to M_1 \to M \to M/M_1 \to 0$ by tensoring with $P^\vee = \mathrm{Hom}(P, k[[u]] \otimes_{\mathbb{F}_p} k_E)^{\mathcal{O}}$ (Construction 4.7.11). Thus we can identify $H = H_1 \oplus H_2$ with Frobenius given by

$$\varphi_H = (\varphi_{H_1} + f \circ \varphi_{H_2}, \varphi_{H_2})$$

for some $f \in \mathrm{Hom}(H_2, H_1)^{\mathcal{O}}$ (see Construction 4.7.1 and Remark 4.7.10). Since H admits a crystalline G_K-action there is a cocycle $\sigma \mapsto f_\sigma$ in $Z^1(G_K, \mathrm{Hom}(H_2, H_1)^{\mathcal{O}})$ such that

$$\sigma_H = (\sigma_{H_1} + f_\sigma \circ \sigma_{H_2}, \sigma_{H_2})$$

The fact that σ_H commutes with φ_H implies $\varphi \circ f_\sigma \circ \varphi^{-1} - f_\sigma = \sigma \circ f \circ \sigma^{-1} - f$.

The map $H_1^{\mathrm{Gal}} \to H^{\mathrm{Gal}}$ is clearly injective. For exactness in the middle, suppose $h \in H^{\mathrm{Gal}}$ is zero in $u^{-p}H_2$ so that $h \in u^{-p}H_1$. Since $h \in H^{\mathrm{Gal}}$ there is a corresponding 1-cocycle h_σ satisfying the conditions of Construction 5.4.3. We have to show there exists such a 1-cocycle contained in $Z^1(G_K, H_1^\flat)$. We can write as $h_\sigma = (h_\sigma^1, h_\sigma^2)$. Since $(\varphi - 1)(h_\sigma^1, h_\sigma^2) = (\sigma - 1)(h, 0)$ and $(h_{\sigma\tau}^1, h_{\sigma\tau}^2) = (h_\sigma^1, h_\sigma^2) + \sigma(h_\tau^1, h_\tau^2)$ we have that: $h_\sigma^2 \in Z^1(G_K, H_2^\flat)$, $\varphi(h_\sigma^2) = h_\sigma^2$ and h_σ^1 satisfies

$$\begin{aligned} h_{\sigma\tau}^1 &= h_\sigma^1 + \sigma(h_\tau^1) + f_\sigma \circ \sigma(h_\tau^2) \\ (\varphi - 1)(h_\sigma^1) + f \circ \varphi(h_\sigma^2) &= (\sigma - 1)(h) \end{aligned} \tag{5.6.5}$$

Since $h_\sigma^2 \in u^{p/p-1}H_2^\flat$ and $f \in \mathrm{Hom}(H_2, H_1)^{\mathcal{O}}$ it follows that $f \circ \varphi(h_\sigma^2) \in u^{p/p-1}H_1^\flat$. The pair P and M_1 are as in Hypothesis 5.3.1 and so Corollary 5.3.6 says there is a unique $X_\sigma \in u^{p/p-1}H_1^\flat$ such that $(\varphi - 1)(X_\sigma) = (f \circ \varphi(h_\sigma^2))$. Then

$$h_\sigma^1 + X_\sigma \in u^{p/p-1}H_1^\flat$$

satisfies all the conditions which ensure $h \in H_1^{\mathrm{Gal}}$, except possibly that $h_\sigma^1 + X_\sigma$ is a 1-cocycle. To show it is note that by the first equation of (5.6.5) it suffices to check that $X_{\sigma\tau} - X_\sigma - \sigma(X_\tau) = -f_\sigma \circ \sigma(h_\tau^2)$. Note that $X_{\sigma\tau} - X_\sigma - \sigma(X_\tau)$ is the unique solution in $u^{p/p-1}H_1^\flat$ to

$$\begin{aligned} (\varphi - 1)(X) &= f \circ \varphi(h_{\sigma\tau}^2) - f \circ \varphi(h_\sigma^2) - \sigma \circ f \circ \varphi(h_\tau^2) \\ &= f \circ \varphi \circ \sigma(h_\tau^2) - \sigma \circ f \circ \varphi(h_\tau^2) \end{aligned} \tag{5.6.6}$$

(for second equality we've used that h_σ^2 is a 1-cocycle). Recall that $\varphi \circ f_\sigma \circ \varphi^{-1} - f_\sigma = \sigma \circ f \circ \sigma^{-1} - f$. Evaluating at $\sigma \circ \varphi(h_\tau^2)$ and using that φ and σ commute, and that $\varphi(h_\tau^2) = h_\tau^2$, we see that (5.6.6) equals $-(\varphi - 1)(f_\sigma \circ \sigma(h_\tau^2))$. Thus $h_\sigma^1 + X_\sigma$ defines a 1-cocycle and $h \in H_1^{\mathrm{Gal}}$. This finishes the proof. ☐

Proof of Proposition 5.6.3. Lemma 5.4.4 implies $\mathrm{Ext}^1_{\mathrm{SD}}(P, M)^{\mathcal{O}} \subset \mathrm{Ext}^1_{\mathrm{Gal}}(P, M)^{\mathcal{O}}$; we have to show this inclusion is an equality. First suppose that M is of rank one. Then we can choose an identification $N :=$

$\operatorname{Hom}(P,M)^{\mathcal{O}} = \overline{\mathfrak{S}}(\{r_\tau\};a)$ with $r_\tau \in [-p,p]$ and $a \in k_E^\times$, and generators e_τ of N_τ such that

$$\varphi(e_{\tau\circ\varphi}) = (a)_\tau u^{r_\tau} e_\tau$$
$$\sigma(e_\tau) = \eta(\sigma)^{\Theta_\tau} e_\tau$$

where $\Theta_\tau = \sum_0^{f-1} r_{\tau\circ\varphi^i} p^i$ (see Lemma 5.1.6). Let $f = \sum F_\tau e_\tau \in N^{\mathrm{Gal}}$ represent a class in $H^1_{\mathrm{Gal}}(N)$. We have to show this class is in $H^1_{\mathrm{SD}}(N)$. Since $H^1_{\mathrm{SD}}(N) \subset H^1_{\mathrm{Gal}}(N)$ and every element of $k_E[[u]]e_\tau$ represents a class in $H^1_{\mathrm{SD}}(N)$ we can assume that

$$F_\tau = F_\tau^{(p)} u^{-p} + \ldots + F_\tau^{(1)} u^{-1}, \qquad F_\tau^{(i)} \in k_E$$

Since f is in N^{Gal} there are $f_\sigma \in u^{p/p-1}N^\flat$ such that $(\varphi-1)(f_\sigma) = (\sigma-1)(f)$. Thus, as $(\varphi-1)(f_\sigma) \in u^{p/p-1}N^\flat$, we have that $(\sigma-1)(f) = \sum(\sigma(F_\tau)\eta(\sigma)^{\Theta_\tau} - F_\tau)e_\tau \in u^{p/p-1}N^\flat$. In other words we have that

$$\sigma(F_\tau)\eta(\sigma)^{\Theta_\tau} - F_\tau \in u^{p/p-1}\mathcal{O}_{C^\flat} \tag{5.6.7}$$

(recall that $\eta(\sigma)$ is a p^f-1-th root of $\epsilon(\sigma) = \sigma(u)/u$). Let $\sigma \in G_K$ be such that $\epsilon(\sigma)$ is a $\mathbb{Z}_p$-generator of $\mathbb{Z}_p(1)$. Lemma 5.6.8 below shows that

$$\sigma(u^i)\eta(\sigma)^{\Theta_\tau} - u^i = u^i(\eta(\sigma)^{\Theta_\tau+(p^f-1)i} - 1)$$

with $-p \le i < 0$ has valuation $i + \frac{p}{p-1}$ unless $i \equiv \Theta_\tau \equiv r_\tau$ modulo p, in which case it has valuation $\ge i + \frac{p^2}{p-1}$. The divisibility (5.6.7) therefore implies $F_\tau^{(i)} = 0$ unless $i = r_\tau$, and so $f = \sum_{r_\tau<0} F_\tau^{(r_\tau)} u^{r_\tau} e_\tau$ which is contained in $\varphi(N)$. We conclude that f represents a class in $H^1_{\mathrm{SD}}(N)$ which proves the proposition when M is of rank one.

LEMMA 5.6.8. *Let $n \in \mathbb{Z}$. Let $\sigma \in G_K$ be such that $\epsilon(\sigma) = \eta(\sigma)^{p^f-1}$ is a $\mathbb{Z}_p$-generator of $\mathbb{Z}_p(1)$. Then the valuation of $\eta(\sigma)^n - 1$ is $\frac{p^{1+v_p(n)}}{p-1}$.*

PROOF. This is [**16**, Lemma 6.6]. □

For the general case of the proposition we argue by induction on the length of M. Since P and M are as in Hypothesis 5.3.1 we can fit M into an exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$ such that P and M_1 are also as in Hypothesis 5.3.1. Combining Lemma 5.6.4 and Remark 4.7.18 we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\operatorname{Hom}_{\mathrm{BK}}(P,M/M_1) & \longrightarrow & \operatorname{Ext}^1_{\mathrm{Gal}}(P,M_1) & \longrightarrow & \operatorname{Ext}^1_{\mathrm{Gal}}(P,M) & \longrightarrow & \operatorname{Ext}^1_{\mathrm{Gal}}(P,M/M_1) & & \\
\| & & \uparrow & & \uparrow & & \uparrow & & \\
\operatorname{Hom}_{\mathrm{BK}}(P,M/M_1) & \longrightarrow & \operatorname{Ext}^1_{\mathrm{SD}}(P,M_1) & \longrightarrow & \operatorname{Ext}^1_{\mathrm{SD}}(P,M) & \longrightarrow & \operatorname{Ext}^1_{\mathrm{SD}}(P,M/M_1) & \longrightarrow & 0
\end{array}$$

By our inductive hypothesis the first and the third inclusions are equalities, so we deduce the middle inclusion is an equality. This completes the proof of the proposition. □

CHAPTER 6

Comparison with Fontaine–Laffaille Theory

As in the previous chapter we assume throughout that $K = K_0$. We describe a functor from the category of objects of $\mathrm{Mod}_k^{\mathrm{SD}}$ with $\mathrm{Weight}(M) \subset [0, p-1]$, to the category of strongly divisible Fontaine–Laffaille modules as considered in [**12**].

When $\mathrm{Weight}(M) \subset [0, p-2]$ we compare the Galois representation associated to the corresponding Fontaine–Laffaille module in [**12**] with the G_{K_∞}-representation $T(M)$. We are unable to do this when $\mathrm{Weight}(M) \subset [0, p-1]$ although we expect the same result to hold.

1. Fontaine-Laffaille Theory

For this section we shall not work with coefficients. Following [**12**] we write MF_k for the category whose objects are triples (M, M^i, φ^i) where M is a k-vector space, M^i is a filtration on M and $\varphi^i : M^i \to M$ is a φ-semilinear map such that $\varphi^i(M^{i+1}) = 0$.

Construction 6.1.1. If $M \in \mathrm{Mod}_k^{\mathrm{BK}}$ is such that $\mathrm{Weight}(M) \subset [0, p-1]$ then we associate to M a triple

$$(M_k, F^i M_k, \varphi^i)$$

in MF_k as follows. The vector space M_k and the filtration on M_k are as defined in Lemma 4.3.2. The φ^i are semilinear maps $F^i M_k \to M_k$ whose construction we now explain. For $i \in \mathbb{Z}$ we have maps $\frac{\varphi}{u^i} : F^i M \to M$ which fit into commutative diagrams

$$\begin{array}{ccc} F^{i-p}M & \xrightarrow{u} & F^i M \\ \Big\downarrow{\scriptstyle \frac{\varphi}{u^{i-p}}} & & \Big\downarrow{\scriptstyle \frac{\varphi}{u^i}} \\ M & = & M \end{array}$$

If $i \in [0, p-1]$ then the image of $\frac{\varphi}{u^{i-p}}$ is contained in uM. Therefore the map $\frac{\varphi}{u^i}$ descends to a semilinear $\varphi^i : F^i M_k = F^i M / u(F^{i-p}M) \to M_k$. Note that $\varphi^i(F^{i+1} M_k) = 0$ and so $M \mapsto (M_k, F^i M_k, \varphi^i)$ describes a functor taking values in MF_k. Observe that this construction only makes sense if $\mathrm{Weight}(M) \subset [0, p-1]$.

Lemma 6.1.2. *With M as above, M is strongly divisible as a Breuil–Kisin module if and only if M_k is a strongly divisible object of MF_k (i.e. $\sum \varphi^i(F^i M_k) = M_k$).*

PROOF. Let (f_i) be a $k[[u]]$-basis of M such that $(u^{r_i}f_i)$ is a $k[[u]]$-basis of $\varphi(M)$, as in Lemma 4.3.4. Let $g_i \in M$ be such that $\varphi(g_i) = u^{r_i}f_i$. Letting $\overline{g}_i, \overline{f}_i$ denote the respective images in M_k, then $\overline{g}_i \in F^{r_i}M_k$ and $\varphi^{r_i}(\overline{g}_i) = \overline{f}_i$. Thus if $M \in \mathrm{Mod}_k^{\mathrm{SD}}$ then $\sum \varphi^i(F^iM_k) = M_k$.

Conversely, choose a basis $(\overline{g}_i)$ of M_k adapted to the filtration, so that there are integers r_j such that $F^iM_k = \bigoplus_{i \geq r_j} k\overline{g}_j$. By construction $F^pM_k = 0$ and $F^0M_k = M_k$ so $r_j \in [0, p-1]$. Choose lifts $g_i \in F^{r_i}M$ of the $\overline{g}_i$ and let $f_i = u^{-r_i}\varphi(g_i) \in M$. Since the g_i form a basis of M the $(u^{r_i}f_i)$ form a basis of $\varphi(M)$. If $\overline{f}_i$ denotes the image of f_i in M_k then $\varphi^{r_i}(\overline{g}_i) = \overline{f}_i$. Therefore, if $\sum \varphi^i(F^iM_k) = M_k$ then the $\overline{f}_i$ form a basis of M_k, and hence the f_i form a basis of M. This shows M is strongly divisible. $\square$

DEFINITION 6.1.3. Let $\mathrm{MF}_k^{\mathrm{SD}}$ denote the full subcategory of MF_k whose objects are finite dimensional over k, are strongly divisible (in the sense that $\sum \varphi^i(M^i) = M$) and satisfy $M^0 = M, M^p = 0$.

2. Comparison with Strong Divisibility

Recall that a (contravariant) functor is defined from $\mathrm{MF}_k^{\mathrm{SD}}$ into the category of finite dimensional $\mathbb{F}_p$-vector spaces equipped with a continuous G_K-action by

$$T^*_{\mathrm{FL}}(M) = \mathrm{Hom}_{\mathrm{MF}_k}(M, \tilde{S})$$

where $\tilde{S} \in \mathrm{MF}_k$ is an object that may be identified with a subring of A_{crys}/p (the functor described in [**12**] is in terms of Ext groups but coincides with T^*_{FL} by [**12**, Lemme 3.8]). The G_K-action on $T^*_{\mathrm{FL}}(M)$ is induced by the G_K-action on $\tilde{S}$. This functor preserves dimensions, i.e. the k-dimension of M equals the $\mathbb{F}_p$-dimension of $T^*_{\mathrm{FL}}(M)$. Explicitly $\tilde{S}$ can be defined as follows (see [**12**, Proposition 5.9]).

- As a k-vector space $\tilde{S} = (\mathcal{O}_C/p)[\xi]$ is a polynomial ring in one variable over $\mathcal{O}_C/p$.
- The filtration $\tilde{S}^i$ is the ideal generated by $p^{i/p}$ and ξ for $0 \leq i < p$, and $\tilde{S}^p = 0$.
- The frobenius $\varphi^i : \tilde{S}^i \to \tilde{S}$ sends

$$\lambda_0 + \lambda_1\xi + \lambda_2\xi^2 + \ldots \mapsto \begin{cases} \varphi^i(\lambda_0)(1+\xi)^i & \text{if } i < p-1 \\ \varphi^i(\lambda_0)(1+\xi)^i + \lambda_1^p(1+\xi)^p & \text{if } i = p-1 \end{cases}$$

  Here $\varphi^i(\lambda_0)$ is the image in $\mathcal{O}_C/p = \mathcal{O}_{C^\flat}/u$ of $\tilde{\lambda}_0^p/u^i$ for any lift $\tilde{\lambda}_0 \in \mathcal{O}_{C^\flat}$ of λ_0.

The G_K-action on $\tilde{S}$ is the natural one on $\mathcal{O}_C/p$ and fixes ξ.

REMARK 6.2.1. Let $\mathfrak{a} = \{x \in \mathcal{O}_C \mid v_p(x) > \frac{p-2}{p-1}\}$. If $M^{p-1} = 0$ then there is a G_K-equivariant map

$$T^*_{\mathrm{FL}}(M) \to \mathrm{Hom}_{\mathrm{MF}_k}(M, \mathcal{O}_C/\mathfrak{a})$$

induced by the map $\tilde{S} \to \mathcal{O}_C/\mathfrak{a}$ sending $\xi \mapsto 0$. In [**9**, 3.3. Lemme 1] it is shown this map is an isomorphism.

PROPOSITION 6.2.2. *Let* $M \in \mathrm{Mod}_k^{\mathrm{SD}}$ *and suppose that* $\mathrm{Weight}(M) \subset [0, p-2]$. *Then there exists functorial* G_{K_∞}*-equivariant identifications*

$$\mathrm{Hom}(T(M), \mathbb{F}_p) = T^*_{\mathrm{FL}}(M_k)$$

PROOF. As the functor in Proposition 2.4.5 is a tensor functor, $\mathrm{Hom}(T(M), \mathbb{F}_p) = \mathrm{Hom}_{\varphi, k[[u]]}(M, C^\flat)$. Let

$$\mathfrak{a}^\flat = \{x \in \mathcal{O}_{C^\flat} \mid v^\flat(x) > \tfrac{p-2}{p-1}\}$$

Lemma 6.2.4 below shows that the image of any $f \in \mathrm{Hom}_{\varphi, k[[u]]}(M, C^\flat)$ is contained in $\mathcal{O}_{C^\flat}$ and that reducing modulo $\mathfrak{a}^\flat$ induces an injection

$$\mathrm{Hom}_{\varphi, k[[u]]}(M, C^\flat) \to \mathrm{Hom}(M_k, \mathcal{O}_{C^\flat}/\mathfrak{a}^\flat)$$

Since the source of this map consists of φ-equivariant homomorphisms one easily checks that the image is contained in $\mathrm{Hom}_{\mathrm{MF}_k}(M_k, \mathcal{O}_C/\mathfrak{a})$ (here we have identified $\mathcal{O}_C/\mathfrak{a} = \mathcal{O}_{C^\flat}/\mathfrak{a}^\flat$ as quotients of $\mathcal{O}_C/p = \mathcal{O}_{C^\flat}/u$).

Since $\mathrm{Weight}(M) \subset [0, p-2]$, $F^{p-1}M_k = 0$ and so by Remark 6.2.1 we have an injective map $\mathrm{Hom}(T(M), \mathbb{F}_p) \to T^*_{\mathrm{FL}}(M_k)$. As the source and target have the same finite $\mathbb{F}_p$-dimension we conclude that this injection is an isomorphism. □

REMARK 6.2.3. The proposition should be true allowing weights in $[0, p-1]$, but we are unsure how to compare $\tilde{S}$ with $\mathcal{O}_{C^\flat}$.

LEMMA 6.2.4. *Let* $M \in \mathrm{Mod}_k^{\mathrm{BK}}$ *be such that* $\mathrm{Weight}(M) \subset [0, i]$. *Then any* φ*-equivariant* $k[[u]]$*-linear homomorphism* $f : M \to C^\flat$ *has image in* $\mathcal{O}_{C^\flat}$. *Further the image is not contained in* $u^{\frac{i}{p-1}+\epsilon}\mathcal{O}_{C^\flat}$ *for any* $\epsilon > 0$.

PROOF. The fact that $\mathrm{Weight}(M) \subset [0, i]$ is equivalent to asking that $u^i M \subset M^\varphi \subset M$. Thus

$$u^i f(M) \subset f(M^\varphi) \subset f(M)$$

As M is finitely generated $f(M)\mathcal{O}_{C^\flat} \subset C^\flat$ is a fractional ideal and so equals $u^\alpha \mathcal{O}_{C^\flat}$ for some $\alpha \in \mathbb{Q}$. Since f is φ-equivariant and φ is an automorphism of $\mathcal{O}_{C^\flat}$ we have that $f(M^\varphi)\mathcal{O}_{C^\flat} = \varphi(f(M)\mathcal{O}_{C^\flat})$ and so $u^i f(M)\mathcal{O}_{C^\flat} \subset \varphi(f(M)\mathcal{O}_{C^\flat}) \subset f(M)\mathcal{O}_{C^\flat}$. These inclusions imply that

$$i + \alpha \geq p\alpha \geq \alpha$$

Hence $0 \leq \alpha \leq i/(p-1)$ which proves the lemma. □

CHAPTER 7

Inertial Weights

We continue to assume that $K = K_0$. Our aim is to study the irreducible objects of $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ (i.e. those with $T(M)$ irreducible as a G_{K_∞}-representation). The main result of this chapter is that if $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ is irreducible and is such that $\mathrm{Weight}_\tau(M)$ is distinct for each τ, then the weights of M coincide with the inertial weights of $T(M)$.

1. Irreducible Objects

Any irreducible representation of G_{K_∞} is induced from a character. In this section we explore the extent with which the analogous statement holds for irreducible $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$. We show that this is the case when $\mathrm{Weight}(M) \subset [0, p-1]$.

Notation 7.1.1. We shall consider both objects $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ and $N \in \mathrm{Mod}_l^{\mathrm{BK}}(\mathcal{O})$ where l/k is a finite extension. Recall that in both cases we can decompose

$$M = \prod_{\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, k_E)} M_\tau, \qquad N = \prod_{\theta \in \mathrm{Hom}_{\mathbb{F}_p}(l, k_E)} N_\theta$$

Throughout this chapter we shall use τ to denote an embedding of $k \to k_E$ and θ to denote an embedding of $l \to k_E$.

Lemma 7.1.2. *Let $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ and assume that $T(M) = \mathrm{Ind}_{L_\infty}^{K_\infty} \chi$ for a character χ. Let $f : \mathfrak{S} \to \mathfrak{S}_L$ be as in Notation 5.2.2 with L/K of degree $n = \dim_{k_E} T(M)$. Then there exists a rank one object N in $\mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$ and an inclusion*

$$M \subset f_* N \tag{7.1.3}$$

inducing an isomorphism after applying T such that:

(1) *Write $N \cong \overline{\mathfrak{S}}_L(\{r_\theta\}; x)$. For each $0 \leq i < n$ define*

$$N^i := \overline{\mathfrak{S}}_L(\{r_\theta^{(i)}\}; x)$$

where $r_\theta^{(i)} = r_{\theta \circ \varphi^{-i[k:\mathbb{F}_p]}}$. Then there exist nonzero $N^{(i)} \subset N^i$ such that for any $\tau : k \to k_E$ and any $\theta : l \to k_E$ with $\theta|_k = \tau$ we have

$$\mathrm{Weight}_\tau(M) = \bigcup_0^{n-1} \mathrm{Weight}_\theta(N^{(i)})$$

(2) *If δ_θ is the smallest non-negative integer such that $u^{\delta_\theta} e_\theta \in M$ under* (7.1.3) *then*

$$p\delta_{\theta\circ\varphi} - \delta_\theta + r_\theta \in \mathrm{Weight}_{\theta|_k}(M)$$

and if $p \neq 2$ then $\delta_\theta \in [0,1]$. If $p = 2$ then $\delta_\theta \in [0,1]$ unless all $r_\theta = 0$ and all $\delta_\theta = 2$.

Remark 7.1.4. The important part of this lemma is the existence of N and the fact that the r_θ and the $\delta_{\theta\circ\varphi} - \delta_\theta + r_\theta$ are contained in $[0,p]$. Besides this (1) is used *only* to deduce Corollary 7.1.8 below.

Remark 7.1.5. Assuming the rank one N exists as in the lemma observe that if we allow ourselves to replace M with M tensored with the rank one object $\overline{\mathfrak{S}}(\{0\}; x^{-1/n}) \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ then we can assume that $N = \overline{\mathfrak{S}}(\{r_\theta\}; 1)$.

Proof of Lemma 7.1.2. The representation $T(M)|_{G_{L_\infty}} = T(f^*M)$ decomposes as $\bigoplus \chi^{(\gamma)}$ with $\gamma \in \mathrm{Gal}(L_\infty/K_\infty) = \mathrm{Gal}(l/k)$. Recall each $\gamma \in \mathrm{Gal}(l/k)$ can be written as $\varphi^{j[k:\mathbb{F}_p]}$ for $j = 0, \dots, n-1$. The composition series

$$T^i := \bigoplus_{j=i}^{n-1} \chi^{(\varphi^{j[k:\mathbb{F}_p]})} \subset T(f^*M)$$

is such that $T^i/T^{i+1} = \chi^{(\varphi^{i[k:\mathbb{F}_p]})}$ and corresponds (via Construction 4.2.4) to a composition series[1]

$$0 = F^n \subset \ldots \subset F^0 = f^*M$$

by objects of $\mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$. Each subquotient F^i/F^{i+1} is of rank one with $T(F^i/F^{i+1}) = \chi^{(\varphi^{i[k:\mathbb{F}_p]})}$. Take $N = F^0/F^1$.

Using Lemma 5.2.3 we obtain a map $M \to f_*N$ which induces the identity after applying T (use the commutativity of the diagram in Lemma 5.2.3). Thus this map is injective with torsion cokernel. By functoriality it is $\mathcal{O}$-equivariant and so $M_\tau \subset (f_*N)_\tau = \prod_{\theta|_k=\tau} N_\theta$. The map $f^*M \to N$ (which we shall call γ_0) can be described explicitly in terms of the inclusion $M \subset f_*N$. On $(f^*M)_\theta = M_{\theta|_k} \to N_\theta$ it is given by

$$\sum_{\theta'|_k=\theta|_k} \alpha_{\theta'} e_{\theta'} \mapsto \alpha_\theta e_\theta, \qquad \alpha_{\theta'} \in k_E[[u]]$$

Let us now describe the filtration F^i on f^*M in terms of the N^i (as described in (1) of the lemma). For each θ we can choose generators e^i_θ of N^i_θ such that frobenius is given by $\varphi(e^i_{\theta\circ\varphi}) = (x)u^{r_{\theta\circ\varphi - i[k:\mathbb{F}_p]}} e^i_\theta$ (with $(x) = x$ or 1 depending on θ). One checks that $T(N^i) = \chi^{(\varphi^{i[k:\mathbb{F}_p]})}$ (using Lemma 4.6.4).

[1] Warning! Even though these filtrations split after applying T they need not be split themselves.

There are surjections $\gamma_i : f^*M \to N^i$ which on $(f^*M)_\theta = M_{\theta|_k}$ are given by the φ-equivariant maps

$$\sum_{\theta'|_k=\theta|_k} \alpha_{\theta'} e_{\theta'} \mapsto \alpha_{\theta\circ\varphi^{-i[k:\mathbb{F}_p]}} e^i_\theta, \qquad \alpha_{\theta'} \in k_E[[u]]$$

If we set $F^i = \ker(\gamma_i : F^{i-1} \to N^{i-1})$ then we obtain a filtration with $F^i/F^{i+1} \subset N^i$. As we have already mentioned $T(N^i) = \chi^{(\varphi^{i[k:\mathbb{F}_p]})}$ and so the F^i are the F^i from the first paragraph. Now (1) follows with $N^{(i)} = F^i/F^{i+1}$ because by Proposition 4.4.5.

$$\mathrm{Weight}_\theta(f^*M) = \bigcup_0^{n-1} \mathrm{Weight}_\theta(F^i/F^{i+1})$$

and because $\mathrm{Weight}_{\theta|_k}(M) = \mathrm{Weight}_\theta(f^*M)$ by Lemma 5.2.6.

For (2) let $P \subset N$ be the rank one object of $\mathrm{Mod}^{\mathrm{BK}}_l(\mathcal{O})$ generated by the $u^{\delta_\theta} e_\theta$, so that $f_*P \subset M \subset f_*N$. The integers $p\delta_{\theta\circ\varphi} - \delta_\theta + r_\theta$ are the weights of P and so to prove that $p\delta_{\theta\circ\varphi} - \delta_\theta + r_\theta \in [0,p]$ it suffices to exhibit P as a submodule of f^*M with torsionfree cokernel (because this will imply the weights of P will be contained in $\mathrm{Weight}(f^*M) \subset [0,p]$ by Proposition 4.4.5). The map $P \to f^*M$ described by

$$u^{\delta_\theta} e_\theta \mapsto (u^{\delta_\theta} e_\theta) \otimes i_\theta$$

(where i_θ is the idempotent of $l[[u]] \otimes_{\mathbb{F}_p} k_E$ as in the proof of Lemma 5.2.6) is φ-equivariant and so describes an injective morphism in $\mathrm{Mod}^{\mathrm{BK}}_l(\mathcal{O})$. The cokernel of this map would be torsion if and only if $u^{\delta_\theta - 1} e_\theta \in M$ for some θ, which contradicts the choice of δ_θ. This implies $p\delta_{\theta\circ\varphi} - \delta_\theta + r_\theta \in \mathrm{Weight}_{\theta|_k}(M) \subset [0,p]$. Now we show $\delta_\theta \in [0,1]$. By (1) we know $r_\theta \in [0,p]$ and so $p\delta_{\theta\circ\varphi} - \delta_\theta \leq p$. For any θ we have
(7.1.6)

$$(p^{n[k:\mathbb{F}_p]-1} - 1)\delta_\theta = \sum_{i=0}^{n[k:\mathbb{F}_p]-1} p^i(p\delta_{\theta\circ\varphi^{i+1}} - \delta_{\theta\circ\varphi^i}) \leq p(p^{n[k:\mathbb{F}_p]-1} - 1)/(p-1)$$

As $\delta_\theta \geq 0$ this implies $\delta_\theta \in [0,1]$ unless $p = 2$, in which case we deduce that $\delta_\theta \in [0,2]$. If $p = 2$ and $\delta_{\theta\circ\varphi} = 2$ for some θ then $2\delta_{\theta\circ\varphi} - \delta_\theta + r_\theta = 4 - \delta_\theta + r_\theta \in [0,2]$. Since $r_\theta \in [0,2]$ it follows that $\delta_\theta = 2$ and $r_\theta = 0$. From this we deduce that $\delta_\theta = 2$ and $r_\theta = 0$ for all θ. □

Remark 7.1.7. The map $\gamma_0 : f^*M \to N$ from the above proof is surjective. Since $f_*(uN) \subset M$ it follows that for each θ' with $\theta'|_k = \tau$ there are $\alpha_\theta \in k_E$ such that

$$e_{\theta'} + \sum_{\substack{\theta|_k=\tau \\ \theta\neq\theta'}} \alpha_\theta e_\theta \in M_\tau$$

Corollary 7.1.8. *With notation as in Lemma 7.1.2, if* $r \in \mathrm{Weight}_\tau(M)$ *then there is a* θ *with* $\theta|_k = \tau$ *and an* r_θ *such that*

$$r \in \{r_\theta, r_\theta - 1, r_\theta + p - 1, r_\theta + p\}$$

Proof. From (1) of Lemma 7.1.2 we know $r \in \mathrm{Weight}_\theta(N^{(i)})$ for some i. Since $N^{(i)} \subset N^i$ are both rank ones, if e^i_θ is a generator of N^i_θ then there is a $\delta^i_\theta \geq 0$ such that $u^{\delta^i_\theta} e^i_\theta$ is a generator of $N^{(i)}_\theta$. As the θ-th weight of N^i is r_ι where $\iota = \theta \circ \varphi^{-if}$, we see that $\mathrm{Weight}_\theta(N^{(i)}) = \{p\delta^i_{\theta\circ\varphi} - \delta^i_\theta + r_\iota\}$. Arguing as in (7.1.6) shows that $\delta^i_\theta \in [0, 1]$ for each θ, unless $p = 2$, all the $r_\iota = 0$ and all the $\delta^i_\theta = 2$. The result follows. □

Lemma 7.1.9. *Let* M *be as in Lemma 7.1.2 and assume* $T(M)$ *is irreducible. Then for any* $\sum \alpha_\theta e_\theta \in M$ *with* $\alpha_\theta \in k_E$ *and any* $i \in \mathbb{Z}/p$ *one has*

$$\sum_{r_\theta \equiv i} \alpha_\theta e_\theta \in M$$

Proof. Since M is strongly divisible and irreducible it admits a unique crystalline G_K-action (Proposition 5.5.3). Likewise f_*N admits a unique crystalline G_K-action, and since $M \to f_*N$ induces the identity on $T(M)$ this inclusion must be compatible with the two G_K-actions (Remark 5.1.5). As in Lemma 5.1.6 the G_K-action on f_*N can be described by

$$\sigma(e_\theta) = \eta(\sigma)^{\Theta_\theta} e_\theta, \qquad \Theta_\theta = \sum_{i=0}^{[k:\mathbb{F}_p]n-1} r_{\theta\circ\varphi^i} p^i$$

(recall we use f to denote $[k : \mathbb{F}_p]$). Thus for any $\sum \alpha_\theta e_\theta \in M$ with $\alpha_\theta \in k_E$ one has

$$(\sigma - 1)(\sum \alpha_\theta e_\theta) = \sum \alpha_\theta(\eta(\sigma)^{\Theta_\theta} - 1) e_\theta \in M \otimes_{k[[u]]} u^{p/p-1}\mathcal{O}_{C^\flat}$$

Choose σ such that $\epsilon(\sigma)$ is a $\mathbb{Z}_p$-generator of $\mathbb{Z}_p(1)$, so that $v^\flat(\eta(\sigma)^i - 1) = p^{v_p(i)+1}/(p-1)$ (see Lemma 5.6.8). Then

$$\sum \alpha_\theta \left(\frac{\eta(\sigma)^{\Theta_\theta} - 1}{\eta(\sigma) - 1} \right) e_\theta \in M \otimes_{k[[u]]} \mathcal{O}_{C^\flat}$$

Since $\frac{\eta(\sigma)^{\Theta_\theta}-1}{\eta(\sigma)-1} = 1 + \eta(\sigma) + \ldots + \eta(\sigma)^{\Theta_\theta - 1} \equiv \Theta_\theta = r_\theta$ modulo $u^{p/p-1}\mathcal{O}_{C^\flat}$ and since each $ue_\theta \in M$ we deduce that

$$\sum \alpha_\theta e_\theta \in M \Rightarrow \sum \alpha_\theta r_\theta e_\theta \in (M \otimes_{k[[u]]} \mathcal{O}_{C^\flat}) \cap N = M \tag{7.1.10}$$

whenever $\alpha_\theta \in k_E$. The equality $(M \otimes_{k[[u]]} \mathcal{O}_{C^\flat}) \cap N = M$ follows because $k[[u]] \to \mathcal{O}_{C^\flat}$ is faithfully flat (see e.g. [**5**, §3.5 Proposition 9]). It is straightforward to check that the implication in (7.1.10) is equivalent to asking that for each $i \in \mathbb{Z}/p$

$$\sum \alpha_\theta e_\theta \in M \Rightarrow \sum_{r_\theta \equiv i} \alpha_\theta e_\theta \in M$$

whenever $\alpha_\theta \in k_E$. $\square$

REMARK 7.1.11. With notation as in Lemma 7.1.2, the representation $T(M)$ is irreducible if and only if $\chi^{(\varphi^{fi})} \neq \chi$ for any $0 < i < n$. Equivalently $T(N^i)$ and $T(N)$ are not isomorphic, which is equivalent to asking that for any θ

$$\sum_0^{n-1} r_{\theta\circ\varphi^j} p^j \not\equiv \sum_0^{n-1} r_{\theta\circ\varphi^{j-i[k:\mathbb{F}_p]}} p^j \text{ modulo } p^{[k:\mathbb{F}_p]n} - 1$$

(see Corollary 4.6.5). In particular, for any $0 < i < n$, we cannot have

$$r_\theta = r_{\theta\circ\varphi^{fi}}$$

for every $\theta : l \to k_E$.

We conclude this section by showing that in the Fontaine-Laffaille range irreducible objects of $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ are always induced from rank ones.

PROPOSITION 7.1.12. *Suppose that $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ and suppose that $T(M)$ is irreducible. Then M is as in Lemma 7.1.2 and there is an inclusion $M \subset f_*N$. If* $\mathrm{Weight}(M) \subset [0, p-1]$ *then $M = f_*N$.*

PROOF. Suppose that $M \neq f_*N$, so that not all the δ_θ (as in Lemma 7.1.2(2)) are zero. If all the $\delta_\theta = 1$ then $p\delta_{\theta\circ\varphi} - \delta_\theta + r_\theta = p - 1 + r_\theta$ for all θ. Since these numbers are contained in $\mathrm{Weight}(M)$ we must have all the $r_\theta = 0$. But this contradicts the assumption that M is irreducible (by Remark 7.1.11). If $p = 2$ this also rules out the possibility of $\delta_\theta = 2$ for all θ. Thus there must be θ such that $\delta_{\theta\circ\varphi} = 1$ and $\delta_\theta = 0$. However then $p\delta_{\theta\circ\varphi} - \delta_\theta + r_\theta = p + r_\theta \in [0, p-1]$ which is impossible. $\square$

COROLLARY 7.1.13. *Let $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ be such that $T(M)$ is irreducible. If* $\mathrm{Weight}(M) \subset [0, p-1]$ *then $M = M(\mathrm{Ind}_L^K T)/\varpi$ for some rank one crystalline $\mathcal{O}$-lattice for L, an unramified extension of K. Moreover we can take T so that* $\mathrm{Weight}_\tau(M) = \mathrm{HT}_\tau(\mathrm{Ind}_L^K T)$.

PROOF. Proposition 7.1.12 implies $M = f_*N$ with $N \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ of rank one. After Proposition 4.6.6 there is a rank one crystalline $\mathcal{O}$-lattice T with $\mathrm{HT}_\theta(T) = \mathrm{Weight}_\theta(N)$ and $M(T)/\varpi \cong N$. Now $\mathrm{Ind}_L^K T$ is a crystalline $\mathcal{O}$-lattice with $\mathrm{HT}_\tau(\mathrm{Ind}_L^K T) = \mathrm{HT}_\theta(T)$ for any $\theta : l \to k$ with $\theta|_k = \tau$. (see [**15**, Corollary 7.1.2]). Since $T(f_*M(T)) \cong \mathrm{Ind}_L^K T$ we must have $M(\mathrm{Ind}_L^K T) \cong f_*M(T)$. Hence $M \cong f_*N \cong M(\mathrm{Ind}_L^K T)/\varpi$ and so $\mathrm{Weight}_\tau(M) = \mathrm{HT}_\tau(\mathrm{Ind}_L^K T)$. $\square$

2. Irreducible Objects II

Proposition 7.1.12 is not true without the assumption that $\mathrm{Weight}(M) \subset [0, p-1]$ (see Example 7.2.11). In this section we show however that a slightly weaker result does hold in general. Regrettably our proof is not very elegant.

THEOREM 7.2.1. *Let* $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ *be such that* $T(M)$ *is irreducible of* k_E*-dimension* n *and let* L/K *be the unramified extension of degree* n*. Assume each* $\mathrm{Weight}_\tau(M)$ *consists of distinct integers. Then (with notation as in Notation 5.2.1) there exist rank one objects* $M' \subset N \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$ *such that* $\mathrm{Weight}_\tau(M) = \mathrm{Weight}_\tau(f_*M')$ *and such that there are inclusions*

$$\begin{array}{ccc} f_*M' & & \\ & \searrow & \\ & & f_*N \\ & \nearrow & \\ M & & \end{array}$$

inducing identifications $T(M) = T(f_*M') = T(N)$.

The rank one $N \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$ is obtained from Lemma 7.1.2. In order to prove the theorem we may replace M by the 'unramified twist' as discussed in Remark 7.1.5; thus we can assume $N = \prod k_E[[u]]e_\theta$ with

$$\varphi(e_{\theta\circ\varphi}) = u^{r_\theta} e_\theta$$

We prove the theorem by distinguishing between the following two cases. Recall the integers δ_θ defined in (2) of Lemma 7.1.2.

- Case 1: Each $\delta_\theta \in [0,1]$ and there is a θ' such that $\delta_{\theta'} = 0$ (i.e. $e_{\theta'} \in M$).
- Case 2: All the $\delta_\theta = 1$ (i.e. no $e_\theta \in M$).

Note that these two cases cover all possible situations, even when $p = 2$, since if $\delta_\theta \notin [0,1]$ then all the $r_\theta = 0$ (see (2) of Lemma 7.1.2) which contradicts the irreducibility of $T(M) = T(f_*N)$.

REMARK 7.2.2. Let $M \in \mathrm{Mod}_k^{\mathrm{SD}}$. In this section we will repeatedly use that

- if $m \in M$ then $\varphi(m) \in M$,
- if $m \in M$ is such that $\varphi(m) \in u^{p+1}M$ then $m \in uM$.

To see this note that if $\mathrm{gr}^i(M_k) = 0$ for $i < 0$ then $F^0M_k = F^{-1}M_k = \ldots = M_k$ and so $F^0M = M$, i.e. if $m \in M$ then $\varphi(m) \in M$. If $\mathrm{gr}^i(M_k) = 0$ for $i > p$ then $F^{p+1}M_k = F^{p+2}M_k = \ldots = 0$ and so $F^{p+1}M \subset uM$, i.e if $\varphi(m) \in u^{p+1}M$ then $m \in uM$.

PROOF OF THEOREM 7.2.1 IN CASE 2. Suppose all the $\delta_\theta = 1$. As r_θ and the $p\delta_{\theta\circ\varphi} - \delta_\theta + r_\theta = p - 1 + r_\theta$ are contained in $[0,p]$ we must have $r_\theta \in [0,1]$. We shall show this contradicts the fact that $T(M)$ is irreducible.

For any $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, k_E)$ observe that there cannot exist only one $\iota \in \mathrm{Hom}_{\mathbb{F}_p}(l, k_E)$ with $\iota|_k = \tau$ and $r_\iota = 1$. Indeed by Remark 7.1.7 there exists $\alpha_\theta \in k_E$ such that $e_\iota + \sum \alpha_\theta e_\theta \in M$, and by Lemma 7.1.9, if all the $r_\theta = 0$ then $e_\iota \in M$ which contradicts the fact that $\delta_\iota = 1$. Similarly there cannot exist only one ι with $\iota|_k = \tau$ and $r_\iota = 0$. As a consequence we see that $\dim_{k_E} T(M) \geq 4$; if not then all the r_θ with fixed $\theta|_k$ would have to be equal which contradicts the assumption that $T(M)$ is irreducible (by Remark 7.1.11).

From Corollary 7.1.8 it follows that any weight of M must be contained in the set

$$\{0, 1, p-1, p\}$$

As $\mathrm{Weight}_\tau(M)$ is assumed to consist of distinct integers we must have $\dim_{k_E} T(M) \leq 4$, so $T(M)$ must be 4-dimensional. For any τ and any θ with $\theta|_k = \tau$ the list $(r_{\theta\circ\varphi^{3[k:\mathbb{F}_p]}}, r_{\theta\circ\varphi^{2[k:\mathbb{F}_p]}}, r_{\theta\circ\varphi^{[k:\mathbb{F}_p]}}, r_\theta)$ consists of 0's and 1's. The previous paragraph explained why there cannot be only one 1 or only one 0 in this list. Thus, up to cycling this list i.e. replacing θ with $\theta\circ\varphi^{i[k:\mathbb{F}_p]}$, we can assume that $(r_{\theta\circ\varphi^{3[k:\mathbb{F}_p]}}, r_{\theta\circ\varphi^{2[k:\mathbb{F}_p]}}, r_{\theta\circ\varphi^{[k:\mathbb{F}_p]}}, r_\theta)$ equals one of

$$(0,0,1,1) \quad (0,1,0,1) \quad (1,1,1,1) \quad (0,0,0,0)$$

If one of the last three possibilities occurs for every θ then we would always have $r_\theta = r_{\theta\circ\varphi^{2[k:\mathbb{F}_p]}}$, which contradicts the irreducibility of $T(M)$ by Remark 7.1.11. Thus there must be a τ and θ with $\theta|_k = \tau$ such that the first possibility occurs. Combining Remark 7.1.7 and Lemma 7.1.9 shows there will be an $\alpha_\theta \in k_E$ such that

$$e_{\theta\circ\varphi^{[k:\mathbb{F}_p]}} + \alpha_\theta e_\theta \in M$$

Since φ sends $u(e_{\theta\circ\varphi^{[k:\mathbb{F}_p]+1}} + \alpha_\theta e_{\theta\circ\varphi})$ onto $u^{p+1}(e_{\theta\circ\varphi^{[k:\mathbb{F}_p]}} + \alpha_\theta e_\theta)$ we deduce from Remark 7.2.2 that $e_{\theta\circ\varphi^{[k:\mathbb{F}_p]+1}} + \alpha_\theta e_{\theta\circ\varphi} \in M$. Therefore we must have $r_{\theta\circ\varphi^{[k:\mathbb{F}_p]+1}} = r_{\theta\circ\varphi}$ (by Lemma 7.1.9) and so

$$(r_{\theta\circ\varphi^{3[k:\mathbb{F}_p]+1}}, r_{\theta\circ\varphi^{2[k:\mathbb{F}_p]+1}}, r_{\theta\circ\varphi^{[k:\mathbb{F}_p]+1}}, r_{\theta\circ\varphi}) = (0,0,1,1) \quad \text{or} \quad (1,1,0,0)$$

But this is impossible because arguing inductively it implies that

$$(r_\theta, r_{\theta\circ\varphi^{3[k:\mathbb{F}_p]}}, r_{\theta\circ\varphi^{2[k:\mathbb{F}_p]}}, r_{\theta\circ\varphi^{[k:\mathbb{F}_p]}}) = (0,0,1,1) \quad \text{or} \quad (1,1,0,0)$$

Thus M as in Case 2 cannot exist. □

Remark 7.2.3. The following proof of the theorem in Case 1 will not use that $T(M)$ is irreducible or that the weights of M are distinct. Thus to remove the distinctness of the weights one only has to treat Case 2. However without the distinctness assumption one cannot reduce to the ≤ 4 dimensional situation, in which case the combinatorics seem to become quite complicated.

Proof in Case 1. Suppose $M \neq f_*N$ so that not all the $\delta_\theta = 0$. Then we can choose θ' such that $\delta_{\theta'} = 0$ and $\delta_{\theta'\circ\varphi} = 1$. For $m \in \{0, \ldots, [k : \mathbb{F}_p]n-1\}$ define $\theta_m = \theta'\circ\varphi^m$. Inductively define subsets of $\mathrm{Hom}_{\mathbb{F}_p}(l, k_E)$ by setting $X_{-1} = \emptyset$ and for $0 \leq m \leq n[k:\mathbb{F}_p]-1$ setting $X_m = X_{m-1}$ if there exist $\alpha_\theta \in k_E$ such that

$$e_{\theta_m} + \sum_{\theta\in X_{m-1}} \alpha_\theta e_\theta \in M \tag{7.2.4}$$

Otherwise set $X_m = X_{m-1} \cup \{\theta_m\}$. Thus $X_0 = \emptyset$ and $X_1 = \{\theta_1\}$. Note that $X_{-1} \subset \ldots \subset X_{nf-1} =: X$. Observe that:

(Ob1) If there exists $\alpha_\theta \in k_E$ such that

$$\sum_{\theta \in X} \alpha_\theta e_\theta \in M$$

then all $\alpha_\theta = 0$.

This implies the α_θ from (7.2.4) are unique. Combining this with Lemma 7.1.9 we deduce:

(Ob2) If $\theta_m \notin X$ then there exists unique $\alpha_\theta \in k_E$ such that

$$e_{\theta_m} + \sum_{\substack{\theta \in X_{m-1} \\ r_{\theta_m} \equiv r_\theta \bmod p}} \alpha_\theta e_\theta \in M$$

and such that if $\tau = \theta_m|_k$ then all the θ may be taken to run over θ satisfying $\theta|_k = \tau$. In particular the sum lies in M_τ.

Let us now outline the strategy for the rest of the proof. Let $M' \subset N$ be the rank one object with $(M')_\theta$ generated by

$$f_\theta = \begin{cases} e_\theta & \text{if } \theta \notin X \\ ue_\theta & \text{if } \theta \in X \end{cases}$$

Setting $g_{\theta\circ\varphi} = f_{\theta\circ\varphi}$ (the reason for this notation should become clear after reading the below) observe that we have

$$\varphi(g_{\theta\circ\varphi}) = u^{p+r_\theta} f_\theta \qquad \text{if } \theta \notin X, \theta\circ\varphi \in X \tag{C1}$$

$$\varphi(g_{\theta\circ\varphi}) = u^{r_\theta} f_\theta \qquad \text{if } \theta, \theta\circ\varphi \notin X \tag{C2}$$

$$\varphi(g_{\theta\circ\varphi}) = u^{r_\theta - 1} f_\theta \qquad \text{if } \theta \in X, \theta\circ\varphi \notin X \tag{C3}$$

$$\varphi(g_{\theta\circ\varphi}) = u^{r_\theta + p - 1} f_\theta \qquad \text{if } \theta, \theta\circ\varphi \in X \tag{C4}$$

We claim that M' satisfies the requirements of the lemma. In other words we claim that $\mathrm{Weight}_\tau(M) = \cup_{\theta|_k = \tau} \mathrm{Weight}_\theta(M')$. As $\mathrm{Weight}_\tau(M) \subset [0,p]$ we must have $r_\theta = 0$ when θ is as in (C1) and $r_\theta > 0$ when θ is as in (C3). This is the content of the next two lemma's (note we should also have $0 \leq r_\theta \leq 1$ when θ is as in (C4) but in this case $\delta_{\theta\circ\varphi} = \delta_\theta = 1$ and so this follows from the fact that $p\delta_{\theta\circ\varphi} - \delta_\theta + r_\theta \in [0,p]$).

Lemma 7.2.5. *If $\theta \in X$ and $\theta \circ \varphi \notin X$ then $r_\theta > 0$.*

Proof. Let $\theta = \theta_m$ and let us use θ to denote a general element of $\mathrm{Hom}_{\mathbb{F}_p}(l, k_E)$. If $m = fn-1$ then $p\delta_{\theta_0} - \delta_{\theta_{fn-1}} + r_{\theta_{fn-1}} = r_{\theta_{[k:\mathbb{F}_p]n-1}} - 1 \in [0,p]$ by (2) of Lemma 7.1.2, and so $r_{\theta_{[k:\mathbb{F}_p]n-1}} > 0$ (note: for this to be true we require $e_{\theta_0} = e_{\theta'} \in M$; this is the point we use that we are in Case 1). If $m \neq [k : \mathbb{F}_p]n - 1$ then by (Ob2) there are $\alpha_{\theta\circ\varphi} \in k_E$ such that

$e_{\theta_{m+1}} + \sum_{\theta\circ\varphi\in X_m} \alpha_{\theta\circ\varphi} e_{\theta\circ\varphi} \in M$.[2] If $r_{\theta_m} = 0$ then applying φ (Remark 7.2.2) we see that

$$e_{\theta_m} + \sum_{\theta\circ\varphi\in X_m} u^{r_\theta} \alpha_{\theta\circ\varphi} e_\theta \in M$$

As $uN \subset M$ we can remove the terms with $r_\theta > 0$ from this sum. Thus we may suppose $e_{\theta_m} + \sum_{\theta\circ\varphi\in X_m} \alpha_{\theta\circ\varphi} e_\theta \in M$. Each θ appearing in this sum is either in X_{m-1} or is not, in either case we can write $e_\theta + \sum_{\iota\in X_{m-1}} \beta_\iota e_\iota \in M$ for some $\beta_\iota \in k_E$. It follows there are $\gamma_\kappa \in k_E$ such that

$$e_{\theta_m} + \sum_{\kappa\in X_{m-1}} \gamma_\kappa e_\kappa \in M$$

which contradicts the fact that $\theta_m \in X$. $\square$

LEMMA 7.2.6. *If $\theta \notin X$ and $\theta\circ\varphi \in X$ then $r_\theta = 0$.*

PROOF. Again let $\theta = \theta_m$ and again let us use θ to denote a general element of $\mathrm{Hom}_{\mathbb{F}_p}(l, k_E)$. Because $e_{\theta_{m+1}} \notin M$ we have that $p\delta_{\theta_{m+1}} - \delta_{\theta_m} + r_{\theta_m} = p - \delta_{\theta_m} + r_{\theta_m}$ (note that $m \neq n[k:\mathbb{F}_p] - 1$ since $e_{\theta'} \in M$). By (2) of Lemma 7.1.2 this is contained in $[0,p]$ so $r_{\theta_m} \in [0,1]$. Suppose $r_{\theta_m} = 1$. Using (Ob2) there are $\alpha_\theta \in k_E$ such that

$$e_{\theta_m} + \sum_{\substack{\theta\in X_{m-1}\\ r_\theta = 1}} \alpha_\theta e_\theta \in M$$

Here we have used that if $r_\theta \equiv 1$ modulo p and in $[0,p]$ then $r_\theta = 1$. Using Remark 7.2.2 we deduce that

$$e_{\theta_{m+1}} + \sum_{\theta\in X_{m-1}} \alpha_\theta e_{\theta\circ\varphi} \in M$$

(because u times this element is in M and φ of it equals u^{p+1} times the element above it). For each $e_{\theta\circ\varphi}$ appearing in this sum, $\theta\circ\varphi$ is either an element of X_m or is such that $e_{\theta\circ\varphi} + \sum_{\iota\in X_m} \beta_\iota e_\iota \in M$ for some $\beta_\iota \in k_E$. It therefore follows that there exists $\gamma_\kappa \in k_E$ such that $e_{\theta_{m+1}} + \sum_{\kappa\in X_m} \gamma_\kappa e_\kappa \in M$ which contradicts the assumption that $\theta_{m+1} \in X$. $\square$

To prove the theorem we have to show $\mathrm{Weight}_\tau(M) = \mathrm{Weight}_\tau(f_* M')$. Using Lemma 4.4.4 and Remark 4.4.7 we see this is implied by the following claim.

CLAIM. For each τ there are $k_E[[u]]$-bases $(g_{\theta\circ\varphi})_{\theta|_k=\tau}$ of $M_{\tau\circ\varphi}$ and $(f_\theta)_{\theta|_k=\tau}$ of M_τ such that the identities (C1)-(C4) hold.

We define the $g_{\theta\circ\varphi}$ and f_θ case-by-case.

[2] If $m = [k:\mathbb{F}_p]n - 1$ then θ_{m+1} doesn't make sense, which is why we had to treat that case separately.

- Consider (C1), i.e. $\theta \notin X$ and $\theta \circ \varphi \in X$. Lemma 7.2.6 implies $r_\theta = 0$. By (Ob2) there are $\alpha_\iota \in k_E$ such that
$$f_\theta := e_\theta + \sum_{\iota \in X} \alpha_\iota e_\iota \in M_\tau$$
with $r_\iota \equiv 0$ modulo p. If $\iota \circ \varphi \notin X$ then $r_\iota > 0$ (by Lemma 7.2.5), so $r_\iota = p$. On the other hand if $e_{\iota\circ\varphi} \notin M$ then $r_\iota \in [0,1]$ (because $p\delta_{\iota\circ\varphi} - \delta_\iota + r_\iota = p - \delta_\iota + r_\iota \in [0,p]$). We deduce that if $\iota \circ \varphi \notin X$ then $e_{\iota\circ\varphi} \in M$. If instead $\iota \circ \varphi \in X$ then $r_\iota \in [0,1]$ (because $p\delta_{\iota\circ\varphi} - \delta_\iota + r_\iota = p - \delta_\iota + r_\iota \in [0,p]$) and so we must have $r_\iota = 0$. Therefore
$$g_{\theta\circ\varphi} := ue_{\theta\circ\varphi} + \sum_{\substack{r_\iota = 0 \\ \iota\circ\varphi \in X}} u\alpha_\iota e_{\iota\circ\varphi} + \sum_{\substack{r_\iota = p \\ e_{\iota\circ\varphi} \in M}} \alpha_\iota e_{\iota\circ\varphi} \in M_\tau$$
because $uN \subset M$. We have $\varphi(g_{\theta\circ\varphi}) = u^p f_\theta$.
- Consider (C2), i.e. $\theta \notin X$ and $\theta \circ \varphi \notin X$, and suppose $r_\theta = 0$. Using (Ob2) there are $\alpha_{\iota\circ\varphi} \in k_E$ such that
$$g_{\theta\circ\varphi} := e_{\theta\circ\varphi} + \sum_{\iota\circ\varphi \in X} \alpha_{\iota\circ\varphi} e_{\iota\circ\varphi} \in M_{\tau\circ\varphi}$$
We define $f_\theta := \varphi(g_{\theta\circ\varphi}) \in M_\tau$. Thus
$$f_\theta = e_\theta + \sum_{\iota\circ\varphi \in X} \alpha_{\iota\circ\varphi} u^{r_\iota} e_\iota$$
- Consider (C2), i.e. $\theta \notin X$ and $\theta \circ \varphi \notin X$, and suppose $r_\theta > 0$. By (Ob2) there are $\alpha_\iota \in k_E$ such that
$$f_\theta := e_\theta + \sum_{\iota \in X} \alpha_\iota e_\iota \in M_\tau$$
with each $r_\iota \equiv r_\theta$ modulo p. Define
$$g_{\theta\circ\varphi} := e_{\theta\circ\varphi} + \sum_{r_\iota > 0} \alpha_\iota e_{\iota\circ\varphi} + u \sum_{r_\iota = 0} \alpha_\iota e_{\iota\circ\varphi}$$
with the sums running over the ι appearing in f_θ. Note the last term can appear only if $r_\theta = p$. We have $\varphi(g_{\theta\circ\varphi}) = u^{r_\theta} f_\theta$ and so $\varphi(ug_{\theta\circ\varphi}) = u^{p+r_\theta} f_\theta$. As $r_\theta > 0$ Remark 7.2.2 implies $g_{\theta\circ\varphi} \in M_{\tau\circ\varphi}$.
- Consider (C3), i.e. $\theta \in X$ and $\theta \circ \varphi \notin X$. By Lemma 7.2.5 we know $r_\theta > 0$. Define
$$g_{\theta\circ\varphi} := e_{\theta\circ\varphi} + \sum_{\iota\circ\varphi \in X} \alpha_{\iota\circ\varphi} e_{\iota\circ\varphi} \in M_{\tau\circ\varphi}$$
as in (Ob2) and define
$$f_\theta := ue_\theta + \sum_{\iota\circ\varphi \in X} \alpha_{\iota\circ\varphi} u^{r_\iota} e_\iota$$

Now if $e_{\theta\circ\varphi} \in M$ then by uniqueness of (Ob2), $g_{\theta\circ\varphi} = e_{\theta\circ\varphi}$ and $f_\theta = ue_\theta$ so $\varphi(g_{\theta\circ\varphi}) = u^{r_\theta - 1} f_\theta$. If $e_{\theta\circ\varphi} \notin M$ then $r_\theta = 1$ (because $p\delta_{\theta\circ\varphi} - \delta_\theta + r_\theta = p - 1 + r_\theta \in [0, p]$) and so $\varphi(g_{\theta\circ\varphi}) = f_\theta = u^{r_\theta - 1} f_\theta$. In particular we see that $f_\theta \in M_\tau$.

- Consider (C4), i.e. $\theta \in X$ and $\theta \circ \varphi \in X$. Put $g_{\theta\circ\varphi} := ue_{\theta\circ\varphi}$ and $f_\theta := ue_\theta$. Then $\varphi(g_{\theta\circ\varphi}) = u^{r_\theta + p - 1} f_\theta$.

The next two lemmas verify the claim and so finish the proof. □

LEMMA 7.2.7. *The $(f_\theta)_{\theta|_k = \tau}$ just defined form a $k_E[[u]]$-basis of M_τ.*

PROOF. Let W be the $k_E[[u]]$-span of all the f_θ, so that $W \subset M$. Since the $k_E[[u]]$-rank of M equals nf, the number of the f_θ, it suffices to show $W = M$.

The first step is to show $ue_\theta \in W$ for each θ. If θ is as in (C4) then this is obvious. It is also obvious if θ is as in (C3) and $e_{\theta\circ\varphi} \in M$ for then $f_\theta = ue_\theta$. If θ is as in (C1) then $uf_\theta = ue_\theta + \sum u\alpha_\iota e_\iota$ where the sum runs over $\iota \in X$ such that if $\iota \circ \varphi \notin X$ then $e_{\iota\circ\varphi} \in M$; by the previous two sentences we deduce $ue_\theta \in W$. At this point note we've shown $ue_\iota \in W$ if $\iota \circ \varphi \in X$. If θ is as in (C3) but with $e_{\theta\circ\varphi} \notin M$ then

$$f_\theta = ue_\theta + \sum_{\iota\circ\varphi \in X} \alpha_{\iota\circ\varphi} u^{r_\iota} e_\iota$$

We know the $u^{r_\iota} e_\iota \in W$ when $r_\iota > 0$ so we have that

$$ue_\theta + \sum_{\substack{\iota\circ\varphi \in X \\ r_\iota = 0}} \alpha_{\iota\circ\varphi} e_\iota \in W$$

Split this sum up as

$$ue_\theta + \sum_{\substack{\iota\circ\varphi \in X \\ \iota \in X}} \alpha_{\iota\circ\varphi} e_\iota + \sum_{\substack{\iota\circ\varphi \in X \\ \iota \notin X}} \alpha_{\iota\circ\varphi} e_\iota \in W$$

If $\iota \circ \varphi \in X$ and $\iota \notin X$ then ι is as in (C1) and so $f_\iota = e_\iota + \sum_{\kappa \in X} \alpha_\kappa e_\kappa$. It follows that there are $\beta_\kappa \in k_E$ such that

$$ue_\theta + \sum_{\kappa \in X} \beta_\kappa e_\kappa \in W$$

However then $\sum_{\kappa \in X} \beta_\kappa e_\kappa \in M$ which implies by (Ob1) that all $\beta_\kappa = 0$. Thus $ue_\theta \in W$. At this point we know all $ue_\theta \in W$ except if θ is as in (C2), i.e. $\theta \notin X$ and $\theta \circ \varphi \notin X$. In this case

$$uf_\theta = \begin{cases} ue_\theta + \sum_{\iota \in X} u\alpha_\iota e_\iota & \text{if } r_\theta > 0 \\ ue_\theta + \sum_{\iota\circ\varphi \in X} \alpha_{\iota\circ\varphi} u^{1 + r_\iota} e_\iota & \text{if } r_\theta = 0 \end{cases}$$

and so that $ue_\theta \in W$ follows from all the cases we have previously worked out.

To finish the proof note that if $Q \subset N$ is the k_E-vector space spanned by the e_ι with $\iota \in X$ then (Ob1) implies $Q \cap M = 0$ (we used this above to show $ue_\theta \in W$ when θ is as in (C3)). If θ is as in (C1) then $e_\theta - f_\theta \in Q$ by definition. Using this and the fact that $ue_\theta \in W$ for all θ we see additionally that if θ is as in (C2) with $r_\theta = 0$, then there exists $w \in W$ such that

$$e_\theta - w \in Q$$

This is also true if θ is as in (C2) with $r_\theta > 0$ since then $e_\theta - f_\theta \in Q$. Thus if $\theta \notin X$ there exists $w \in W$ such that $e_\theta - w \in Q$. Now take an arbitrary element $z = \sum \alpha_\theta e_\theta \in M$. We need to show it lies in W. We can assume $\alpha_\theta \in k_E$ because we know $ue_\theta \in W$ for all θ. By the above we can find a $w \in W$ such that $z - w \in Q$; however since $z - w \in M$ we conclude that $z = w$. Thus the (f_θ) generate M which proves the lemma. □

LEMMA 7.2.8. *The $(g_{\theta\circ\varphi})$ just defined form a $k_E[[u]]$-basis of $M_{\tau\circ\varphi}$.*

PROOF. The idea is the same as the previous lemma, but the details are slightly different. Again let $W \subset M$ be the sub-$k_E[[u]]$-module spanned by the $g_{\theta\circ\varphi}$; again we have to show $W = M$.

First we show $ue_{\theta\circ\varphi} \in W$ for all θ. If θ is as in (C4) then this is clear. It is also clear if θ is as in (C3) and $e_{\theta\circ\varphi} \in M$ because in this case $g_{\theta\circ\varphi} = e_{\theta\circ\varphi}$. If θ is as in (C1) then

$$g_{\theta\circ\varphi} = ue_{\theta\circ\varphi} + \sum_{\substack{r_\iota = 0 \\ \iota\circ\varphi \in X}} u\alpha_\iota e_{\iota\circ\varphi} + \sum_{\substack{r_\iota = p \\ e_{\iota\circ\varphi} \in M}} \alpha_\iota e_{\iota\circ\varphi} \in M_\tau$$

with each $\iota \in X$, so using the two previous cases we deduce $ue_{\theta\circ\varphi} \in W$. In particular we've checked $ue_{\theta\circ\varphi} \in W$ whenever $\theta \circ \varphi \in X$. If θ is as in (C3), without $e_{\theta\circ\varphi} \in M$, or as in (C2) with $r_\theta = 0$ then $ue_{\theta\circ\varphi} \in W$ since $ug_{\theta\circ\varphi} = ue_{\theta\circ\varphi} + \sum_{\iota\in X} u\alpha_\iota e_\iota$, and each $ue_\iota \in W$ by the above. At this point the only remaining case is when θ is as in (C2) with $r_\theta > 0$. If θ is as in (C2) with $r_\theta > 0$ then

$$ug_{\theta\circ\varphi} = ue_{\theta\circ\varphi} + \sum_{r_\iota > 0} u\alpha_\iota e_{\iota\circ\varphi} + u^2 \sum_{r_\iota = 0} \alpha_\iota e_{\iota\circ\varphi}$$

with each $\iota \in X$. As we've shown above that if $\iota \in X$ then $ue_{\iota\circ\varphi} \in W$ we deduce that $ue_{\theta\circ\varphi} \in W$. This completes the proof that $ue_\theta \in W$ for all θ.

We finish the proof just as in the previous lemma. Let $Q \subset N$ be the k_E-span of the e_θ with $\theta \in X$, so that $Q \cap M = 0$. If θ is as in (C3) then $e_{\theta\circ\varphi} - g_{\theta\circ\varphi} \in Q$ by construction. Likewise if θ is as in (C2) with $r_\theta = 0$. Using this and the fact that $ue_\theta \in W$ for all θ we also see that if θ is as in (C2) with $r_\theta > 0$ then there exists a $w \in W$ such that $e_{\theta\circ\varphi} - w \in Q$. This shows that if $\theta \circ \varphi \notin X$ then there exists a $w \in W$ such that

$$e_{\theta\circ\varphi} - w \in Q$$

Thus for any general element $Z = \sum \alpha_\theta e_\theta \in M$ there exists $w \in W$ such that $Z - w \in Q \cap M$. We conclude $Z = w \in W$ which finishes the proof. □

In low dimensions there do not exist M as in Theorem 7.2.1 with $M \neq f_* N$:

LEMMA 7.2.9. *Let M be as above but without the assumption that* $\mathrm{Weight}_\tau(M)$ *consists of distinct integers. If $[L:\mathbb{Q}_p]\leq 4$ (i.e. $n[K:\mathbb{Q}_p]\leq 4$) then $M=f_*N$.*

PROOF. Let us treat only the case $[L:\mathbb{Q}_p]=4$, the other cases follow by a similar (and easier) argument.

By the proof of the theorem in case 2 (the arguments in this case when $[L:\mathbb{Q}_p]\leq 4$ are valid without assuming the distinctness of $\mathrm{Weight}_\tau(M)$) we may assume $e_\theta\in M$ and $e_{\theta\circ\varphi}\notin M$. Let us first show that $e_{\theta\circ\varphi^2}\notin M$. If $e_{\theta\circ\varphi^2}\in M$ then by Remark 7.1.7 we must have $e_{\theta\circ\varphi^3}\notin M$ and there must exist an $\alpha\in k_E$ such that

$$e_{\theta\circ\varphi}+\alpha e_{\theta\circ\varphi^3}\in M \tag{7.2.10}$$

Note that $r_{\theta\circ\varphi^3}>0$ since $\delta_{\theta\circ\varphi^3}=1$ and $\delta_\theta=0$. Similarly $r_{\theta\circ\varphi}>0$. Using Lemma 7.1.9 we deduce that $r_{\theta\circ\varphi}=r_{\theta\circ\varphi^3}$. We also have that $r_\theta=0$ since $\delta_\theta=0$ and $\delta_{\theta\circ\varphi}=1$, and likewise $r_{\theta\circ\varphi^2}=0$. This implies $(r_{\theta\circ\varphi^3},r_{\theta\circ\varphi^2},r_{\theta\circ\varphi},r_\theta)=(r,0,r,0)$ where $r=r_{\theta\circ\varphi^3}=r_{\theta\circ\varphi}$, which is periodic of order 2. This contradicts the irreducibility of $T(M)$ by Remark 7.1.11.

Next we show there cannot exist an $\alpha\in k_E$ such that any of $e_{\theta\circ\varphi}+\alpha e_{\theta\circ\varphi^2}, e_{\theta\circ\varphi}+\alpha e_{\theta\circ\varphi^3}$ or $e_{\theta\circ\varphi^2}+\alpha e_{\theta\circ\varphi^3}$ are in M. Suppose $e_{\theta\circ\varphi}+\alpha e_{\theta\circ\varphi^2}\in M$. Since $\delta_{\theta\circ\varphi}=\delta_{\theta\circ\varphi^2}=1$, $r_{\theta\circ\varphi}\in[0,1]$. If $r_{\theta\circ\varphi}=0$ then $\varphi(e_{\theta\circ\varphi}+\alpha e_{\theta\circ\varphi^2})=e_\theta+\alpha e_{\theta\circ\varphi}\in M$ which is impossible. Hence $r_{\theta\circ\varphi}=1$ and as $r_{\theta\circ\varphi}\equiv r_{\theta\circ\varphi^2}$ modulo p we must have $r_{\theta\circ\varphi^2}=1$. But then $\varphi(u(e_{\theta\circ\varphi^2}+\alpha e_{\theta\circ\varphi^3}))=u^{p+1}(e_{\theta\circ\varphi}+\alpha e_{\theta\circ\varphi^2})$ and so by Remark 7.2.2 we have $e_{\theta\circ\varphi^2}+\alpha e_{\theta\circ\varphi^3}\in M$. Let us show this is impossible. If not then we must have $e_{\theta\circ\varphi^3}\notin M$ and so $r_{\theta\circ\varphi^2}\in[0,1]$ and $r_{\theta\circ\varphi^3}>0$. As they are both congruent modulo p they must both be equal to 1. However then $\varphi(u(e_{\theta\circ\varphi^3}+\alpha e_\theta))=u^{p+1}(e_{\theta\circ\varphi^2}+\alpha e_{\theta\circ\varphi^3})$ and so $e_{\theta\circ\varphi^3}+\alpha e_\theta\in M$ which is a contradiction. Similarly one shows there cannot exist $\alpha\in k_E$ such that $e_{\theta\circ\varphi}+\alpha e_{\theta\circ\varphi^3}\in M$.

It follows that there must exist $\alpha_1,\alpha_2\in k_E^\times$ such that $e_{\theta\circ\varphi}+\alpha_1 e_{\theta\circ\varphi^2}+\alpha_2 e_{\theta\circ\varphi^3}\in M$. This is also impossible because then $r_{\theta\circ\varphi}=r_{\theta\circ\varphi^2}=r_{\theta\circ\varphi^3}=1$ and so $e_{\theta\circ\varphi^2}+\alpha_1 e_{\theta\circ\varphi^3}+\alpha_2 e_\theta\in M$ which contradicts the previous paragraph. □

EXAMPLE 7.2.11. The following is an example of M as in Theorem 7.2.1 with $M\neq f_*N$. Let $K=\mathbb{Q}_p$ and L/K be the unramified extension of degree 5. Fix a $\theta\in\mathrm{Hom}_{\mathbb{F}_p}(l,k_E)$ and integers $1\leq n\leq p$, $0\leq x\leq p$. Consider the rank one

$$N=\overline{\mathfrak{S}}(\{x,n,0,n,0\};1)\in\mathrm{Mod}_l^{\mathrm{BK}}(\mathcal{O})$$

Thus $N_{\theta\circ\varphi^i}$ is generated by $e_{\theta\circ\varphi^i}$ and for $i=0,\dots,4$

$$\varphi(e_{\theta\circ\varphi^{i+1}})=u^{r_i}e_{\theta\circ\varphi^i}$$

where $(r_4,\dots,r_0)=(x,n,0,n,0)$. Let $M\subset f_*N$ be the submodule generated by

$$e_{\theta\circ\varphi^4}, e_{\theta\circ\varphi^3}+e_{\theta\circ\varphi}, e_{\theta\circ\varphi^2}, ue_{\theta\circ\varphi}, e_\theta$$

One computes that

$$\varphi(e_{\theta\circ\varphi^4}, e_{\theta\circ\varphi^3}+e_{\theta\circ\varphi}, e_{\theta\circ\varphi^2}, ue_{\theta\circ\varphi}, e_\theta) = (e_{\theta\circ\varphi^4}, e_{\theta\circ\varphi^3}+e_{\theta\circ\varphi}, e_{\theta\circ\varphi^2}, ue_{\theta\circ\varphi}, e_\theta)X$$

where

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u^n & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & u^{n-1} & 0 & 0 \\ 0 & 0 & 0 & u^p & 0 \\ 0 & 0 & 0 & 0 & u^x \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which shows that M is strongly divisible. The pair M and N are as in Lemma 7.1.2 (with notation as in the proof of this lemma one just has to check that the map $\gamma_0 : f^*M \to N$ is surjective, and this is clear). In particular we see that $M \neq f_*(N')$ for *any* rank one $N' \in \mathrm{Mod}_l^{\mathrm{BK}}(\mathcal{O})$. One can take M' from Theorem 7.2.1 to be the Breuil–Kisin module generated by

$$e_{\theta\circ\varphi^4}, e_{\theta\circ\varphi^3}, ue_{\theta\circ\varphi^2}, e_{\theta\circ\varphi}, e_\theta$$

3. Inertial Weights

In this section we show how Theorem 7.2.1 implies Theorem A from the introduction.

COROLLARY 7.3.1. *Let $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ be irreducible with* $\mathrm{Weight}_\tau(M)$ *consisting of distinct integers for each τ. Then there are integers $r_\theta \in [0,p]$ such that, with notation as in Lemma 4.6.4,*

$$T(M) = \mathrm{Ind}_{L_\infty}^{K_\infty}(\psi_a \prod_{\theta : l \to k_E} \chi_\theta^{-r_\theta})$$

and $\mathrm{Weight}_\tau(M) = \{r_\theta \mid \theta|_k = \tau\}$.

PROOF. Theorem 7.2.1 implies $T(M) = T(f_*M')$ with $M' \in \mathrm{Mod}_l^{\mathrm{SD}}(\mathcal{O})$ of rank one, and $\mathrm{Weight}_\tau(M) = \cup_{\theta|_k = \tau} \mathrm{Weight}_\theta(M')$. Thus $T(M) = \mathrm{Ind}_{L_\infty}^{K_\infty} T(M')$. Lemma 4.6.4 implies $T(M') = \psi_a \prod_\theta \chi_\theta^{-r_\theta}$ where $\{r_\theta\} = \mathrm{Weight}_\theta(M')$ so the corollary follows. □

THEOREM 7.3.2. *Let T be a crystalline $\mathcal{O}$-lattice and let $\overline{T} = T/\varpi$. Assume that* $\mathrm{HT}_\tau(T) \subset [0,p]$ *and consists of distinct integers for each τ. Then there are unramified extensions L_ζ/K, with residue field l_ζ, and characters $\zeta : G_{L_\zeta} \to k_E^\times$ such that*

$$\overline{T}^{\mathrm{ss}} \cong \bigoplus_\zeta \mathrm{Ind}_{L_\zeta}^K \zeta \tag{7.3.3}$$

and integers r_θ such that:

(1) *Each induced summand is irreducible.*
(2) *$\zeta = \psi_\zeta \prod \chi_\theta^{-r_\theta}$ where the product runs over $\theta \in \mathrm{Hom}_{\mathbb{F}_p}(l_\zeta, k_E)$ and ψ_ζ denotes some unramified character.*
(3) $\mathrm{HT}_\tau(T) = \{r_\theta \mid \theta|_k = \tau\}$.

PROOF. Our assumption that k_E is sufficiently large means we can write $\overline{T}^{\mathrm{ss}}$ as in (7.3.3). Thus $\overline{T}$ admits a composition series whose subquotients are each isomorphic to $\mathrm{Ind}_{L_\zeta}^K \zeta$. If $M = M(T)/\varpi$ then, since $T(M) = \overline{T}|_{G_{K_\infty}}$, M admits a composition series

$$0 = M_n \subset \ldots \subset M_0 = M$$

such that each $T(M_i/M_{i+1}) \cong \mathrm{Ind}_{L_\zeta}^K \zeta$ for some ζ. If $p = 2$ choose π such that $K_\infty \cap K(\mu_{p^\infty}) = K$ as in [**29**, Lemma 2.1] (recall Remark 4.5.2). Therefore Theorem 4.5.1 implies $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ and $\mathrm{Weight}_\tau(M) = \mathrm{HT}_\tau(T)$. By Proposition 4.4.5 each $M_i/M_{i+1} \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ and

$$\bigcup_i \mathrm{Weight}_\tau(M_i/M_{i+1}) = \mathrm{HT}_\tau(T)$$

In particular $\mathrm{Weight}_\tau(M_i/M_{i+1})$ consists of distinct integers for each i and so Corollary 7.3.1 applies. Thus

$$\mathrm{Ind}_{L_\zeta}^K \zeta|_{G_{K_\infty}} \cong \mathrm{Ind}_{L_{\zeta,\infty}}^{K_\infty} (\psi_a \prod \chi_\theta^{-r_\theta})$$

where the r_θ are such that $\mathrm{Weight}_\tau(M_i/M_{i+1}) = \{r_\theta \mid \theta|_k = \tau\}$. Using that $\mathrm{Ind}_{L_\zeta}^K \zeta|_{G_{K_\infty}} = \mathrm{Ind}_{L_{\zeta,\infty}}^{K_\infty} \zeta$ (see the proof of Lemma 3.2.5) the theorem follows. $\square$

Theorem A is an immediate consequence of Theorem 7.3.2 via a straightforward twisting argument. We conclude this section by showing that the set of inertial weights of a semisimple residual representations $\overline{\rho} : G_K \to \mathrm{GL}_n(k_E)$ depends only on $\overline{\rho}|_{I_K}$.

LEMMA 7.3.4. *Let $\rho, \rho' : G_K \to \mathrm{GL}_n(k_E)$ be two continuous semisimple representations. Suppose there is a finite unramified extension L/K such that there is an isomorphism $\rho|_L \to \rho'|_L$. Then $\rho = \oplus \rho_i$, with each ρ_i irreducible and there exist unramified characters ψ_i such that*

$$\rho' = \bigoplus \rho_i \otimes \psi_i$$

PROOF. The isomorphism $\rho|_L \to \rho'|_L$ induces an isomorphism

$$\iota : \mathrm{Ind}_L^K \rho|_L = \rho \otimes R \to \rho' \otimes R = \mathrm{Ind}_L^K \rho'|_L$$

where $R = \mathrm{Ind}_L^K \mathbb{1}$ is the regular representation for $\mathrm{Gal}(L/K)$. As ρ is semisimple we can write $\rho = \oplus \rho_I$ where each ρ_I is the sum of all those Jordan–Holder factors of ρ which differ by an unramified twist. Likewise we can write $\rho' = \oplus \rho'_J$. The Jordan–Holder factors $\mathrm{JH}(R)$ of R are all unramified characters so ι must restrict, for each I, to maps

$$\rho_I \otimes R \to \rho'_J \otimes R \tag{7.3.5}$$

for some J. Since the direct sum of these restrictions of is an isomorphism each of (7.3.5) is an isomorphism. We can therefore assume $\rho = \rho_I, \rho' = \rho'_J$. The fact that $\mathrm{JH}(R)$ consists of characters means that $\mathrm{JH}(\rho \otimes R) = \mathrm{JH}(\rho) \otimes \mathrm{JH}(R)$, and likewise for $\rho' \otimes R$. This means that every irreducible summand of ρ_I is an irreducible summand of ρ' twisted by some unramified character, and conversely for every irreducible summand of ρ'. From this the lemma follows. $\square$

CHAPTER 8

Crystalline Lifts of Breuil–Kisin modules

We saw in the previous chapter that every crystalline $\mathcal{O}$-lattice gives rise to an object of $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ via $T \mapsto M(T)/\varpi$. In this chapter we explore whether every object of $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ arises in this way. Though we are not able to answer this question in general, in some instances we show it to be true.

1. Crystalline Extensions

The aim of this section is to prove the following proposition.

Proposition 8.1.1. *Let T_i be two crystalline $\mathcal{O}$-lattices with Hodge–Tate weights contained in $[0,p]$. Let $\overline{T}_i = T_i/\varpi$ and suppose that* $\mathrm{Hom}(\overline{T}_2, \overline{T}_1)$ *satisfies the conditions of Proposition 3.3.4. Let $\overline{M}_i = M(T_i)/\varpi$ and consider an exact sequence*

$$0 \to \overline{M}_1 \to M \to \overline{M}_2 \to 0 \tag{8.1.2}$$

representing a class in $\mathrm{Ext}^1_{\mathrm{SD}}(\overline{M}_2, \overline{M}_1)$. *Then there exists an extension $0 \to T_2 \to T \to T_1 \to 0$ of crystalline $\mathcal{O}$-lattices such that $0 \to M(T_2) \to M(T) \to M(T_1) \to 0$ is an exact sequence of Breuil–Kisin modules which after reducing modulo ϖ is the exact sequence* (8.1.2).

In [**16**,**17**] similar results are proven when both T_i are one-dimensional. However their strategy only produces extensions T such that $T(M)$ and $\overline{T}$ are isomorphic as Galois representations, which is weaker than asking that $M(T)/\varpi = M$. As in [**16**,**17**] our proof is based upon a comparison of the dimensions of H^1_{SD} and $\mathrm{Ext}^1_{\mathrm{crys}}$. The argument is complicated by fact that $T \mapsto M(T)$ is not an exact functor. Instead we use that $T \mapsto M(T)$ becomes exact after inverting p.

Notation 8.1.3. Let $\mathrm{Mod}_K^{\text{BK-iso}}(\mathcal{O})$ denote the category of pairs (M,ι) where M is on object of $\mathrm{Mod}_K^{\text{BK-iso}}$ (Definition 2.4.17) and ι is an $\mathcal{O}$-action. Just as in Remark 2.4.18 this is the isogeny category of $\mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$. The functor $V \mapsto M(V)$ of Proposition 2.4.19 sends a crystalline E-representation onto an object of $\mathrm{Mod}_K^{\text{BK-iso}}(\mathcal{O})$ and the bijection of Proposition 2.4.19 restricts to

$$\left\{ \begin{array}{c} \text{Objects } M^\circ \subset M(V) \text{ in } \mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O}) \\ \text{which are } \mathfrak{S}_{\mathcal{O}}\text{-finite free with} \\ M(V) = M^\circ[\frac{1}{p}] \end{array} \right\} \cong \left\{ \begin{array}{c} G_{K_\infty}\text{-stable } \mathcal{O}\text{-lattices} \\ T \subset V \end{array} \right\}$$

This last fact follows because of [**20**, Proposition 2.1.12] which says that $M \mapsto T(M)$ is fully faithful on Breuil–Kisin modules which are finite free over $\mathfrak{S}$.

REMARK 8.1.4. For M in $\mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ or $\mathrm{Mod}_K^{\text{BK-iso}}(\mathcal{O})$ we define $H^i(M)$ as the cohomology of

$$M \xrightarrow{\varphi-1} M[\tfrac{1}{E}]$$

Respectively these are $\mathcal{O}$-modules or E-vector spaces. If $M \in \mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ then the maps $M \to M/\varpi$ and $M \to M[\frac{1}{p}]$ induce maps on cohomology. Since tensor products are right exact $H^1(M) \otimes_{\mathcal{O}} k_E$ is the cokernel of $\varphi - 1 : M/\varpi \to (M/\varpi)[\frac{1}{E}]$ and so the map $H^1(M) \to H^1(M/\varpi)$ induces an isomorphism $H^1(M) \otimes_{\mathcal{O}} k_E \to H^1(M/\varpi)$. Likewise when one applies $\otimes_{\mathcal{O}} E$. Thus we obtain identifications

$$\alpha : H^1(M) \otimes_{\mathcal{O}} E \to H^1(M[\tfrac{1}{p}]), \qquad \beta : H^1(M) \otimes_{\mathcal{O}} k_E = H^1(M/\varpi) \tag{8.1.5}$$

NOTATION 8.1.6. Let $\mathrm{Ext}^1_E(P, M)$ denote the first Yoneda extension group in the abelian category $\mathrm{Mod}_K^{\text{BK-iso}}(\mathcal{O})$. If $\mathrm{Hom}(P, M)^E$ denotes the internal hom in $\mathrm{Mod}_K^{\text{BK-iso}}(\mathcal{O})$, defined in the usual way, then arguing as in Construction 4.7.1 (which is viable since each of P and M are free over $\mathfrak{S} \otimes_{\mathbb{Z}_p} E$, which can be seen using [**4**, Proposition 4.3]) we see there are functorial identifications

$$H^1(\mathrm{Hom}(P, M)^E) = \mathrm{Ext}^1_E(P, M)$$

Likewise if $\mathrm{Ext}^1_{\mathcal{O}}(P^\circ, M^\circ)$ denotes the first Yoneda extension group in $\mathrm{Mod}_K^{\mathrm{BK}}(\mathcal{O})$ then, provided $\mathrm{Hom}(P, M)^{\mathcal{O}}$ is projective as an $\mathfrak{S}_{\mathcal{O}}$-module (e.g. if both P and M are free over $\mathfrak{S}_{\mathcal{O}}$), a variant of Construction 4.7.1 implies that there are functorial identifications

$$H^1(\mathrm{Hom}(P^\circ, M^\circ)^{\mathcal{O}}) = \mathrm{Ext}^1_{\mathcal{O}}(P^\circ, M^\circ)$$

Under our identifications $H^1 = \mathrm{Ext}^1_*$ the maps (8.1.5) are precisely the maps $\mathrm{Ext}^1_E \leftarrow \mathrm{Ext}^1_{\mathcal{O}} \to \mathrm{Ext}^1_{k_E}$ which send (the class of) an exact sequence onto (the class of) that exact sequence tensored repectively with E or k_E

PROOF OF PROPOSITION 8.1.1. Set $V_i = T_i \otimes_{\mathcal{O}} E$ and consider the following diagram.

$$\begin{array}{ccc} & \mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1)) & \xrightarrow{\beta} \mathrm{Ext}^1_{k_E}(\overline{M}_2, \overline{M}_1) \\ & \downarrow \alpha & \\ \mathrm{Ext}^1_{\mathrm{crys}}(V_2, V_1) \xrightarrow{\gamma} & \mathrm{Ext}^1_E(M(V_2), M(V_1)) & \end{array}$$

The maps α and β are those described in (8.1.5). The map γ is obtained by applying $V \mapsto M(V)$ (from Proposition 2.4.19) to exact sequences representing classes in $\mathrm{Ext}_{\mathrm{crys}}(V_2, V_1)$. This makes sense since $V \mapsto M(V)$ is exact. Exactness of $M \mapsto M(V)$ also implies that this functor preserves pushouts and pullbacks; thus γ is E-linear. Since $V \mapsto M(V)$ is fully faithful we see that γ is injective.

Let Θ' denote the image of γ and let Θ denote the preimage of Θ' under α. If $0 \to M(T_1) \to M^\circ \to M(T_2) \to 0$ represents a class in Θ then by definition $M^\circ \otimes_{\mathcal{O}} E = M(V)$ where V is a crystalline E-representation fitting into an extension $0 \to V_1 \to V \to V_2 \to 0$. The bijection from Notation 8.1.3 shows that $T = T(M^\circ)$ is a G_{K_∞}-stable $\mathcal{O}$-lattice inside V which, since $M \mapsto T(M)$ is exact, sits in a G_{K_∞}-equivariant exact sequence $0 \to T_1 \to T \to T_2 \to 0$. Our assumption on $\mathrm{Hom}(\overline{T}_2, \overline{T}_1)$ allows us to apply Proposition 3.3.13; thus T is a G_K-stable lattice in V and so $M^\circ = M(T)$. Using Theorem 4.5.1 it follows that β maps every element of Θ into $\mathrm{Ext}^1_{\mathrm{SD}}(\overline{M}_2, \overline{M}_1)$. Using (8.1.5) we deduce that

$$\Theta \otimes_{\mathcal{O}} k_E \hookrightarrow \mathrm{Ext}^1_{\mathrm{SD}}(\overline{M}_2, \overline{M}_1)$$

On the other hand since α is given by inverting p, the image of α is an $\mathcal{O}$-lattice inside $\mathrm{Ext}^1_E(M(V_2), M(V_1))$ and its kernel is the torsion subgroup $\mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))_{\mathrm{tors}}$. Thus we can decompose Θ as

$$\Theta_{\mathrm{free}} \oplus \mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))_{\mathrm{tors}}$$

where Θ_{free} is a free $\mathcal{O}$-module of rank equal to the E-dimension of $\mathrm{Ext}^1_{\mathrm{crys}}(V_2, V_1)$. Using Lemma 8.1.7 below and Proposition 2.5.7 we see that

$$\dim_{k_E}(\Theta \otimes_{\mathcal{O}} k_E) - \dim_{k_E} \mathrm{Hom}_{\mathrm{BK}}(\overline{M}_2, \overline{M}_1) = \sum_{\tau} \mathrm{Card}(\{i - j < 0 \mid i \in \mathrm{HT}_\tau(V_1), j \in \mathrm{HT}_\tau(V_2)\})$$

Since $\mathrm{HT}_\tau(V_i) = \mathrm{Weight}_\tau(\overline{M}_i)$ by Theorem 4.5.1, it follows from Proposition 4.7.16 that $\mathrm{Ext}^1_{\mathrm{SD}}(\overline{M}_2, \overline{M}_1)$ and $\Theta \otimes_{\mathcal{O}} k_E$ have the same k_E-dimension. Hence

$$\Theta \otimes_{\mathcal{O}} k_E = \mathrm{Ext}^1_{\mathrm{SD}}(\overline{M}_2, \overline{M}_1)$$

which shows that any extension $0 \to \overline{M}_1 \to M \to \overline{M}_2 \to 0$ which represents a class in $\mathrm{Ext}^1_{\mathrm{SD}}(\overline{M}_1, \overline{M}_2)$ arises as the reduction of $0 \to M(T_1) \to M(T) \to M(T_2) \to 0$ for some crystalline extension $0 \to T_1 \to T \to T_2 \to 0$. $\square$

To finish the proof we just need to prove the following lemma.

LEMMA 8.1.7. *If T_1 and T_2 are crystalline $\mathcal{O}$-lattices then*

$$\dim_{k_E}(\mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))_{\mathrm{tors}} \otimes_{\mathcal{O}} k_E) = \dim_{k_E} \mathrm{Hom}_{\mathrm{BK}}(\overline{M}_2, \overline{M}_1) - \dim_E \mathrm{Hom}_{E[G_K]}(V_2, V_1)$$

PROOF. The k_E-dimension of $\mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))_{\mathrm{tors}} \otimes_{\mathcal{O}} k_E$ equals the k_E-dimension of the ϖ-torsion subgroup of $\mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))$. To compute this latter group we consider the exact sequence $0 \to M(T_1) \xrightarrow{\varpi} M(T_1) \to \overline{M}_1 \to 0$ in $\mathrm{Mod}^{\mathrm{BK}}_K(\mathcal{O})$; the associated long exact sequence reads

$$0 \to \mathrm{Hom}_{\mathrm{BK}}(M(T_2), M(T_1)) \xrightarrow{\varpi} \mathrm{Hom}_{\mathrm{BK}}(M(T_2), M(T_1)) \to \mathrm{Hom}_{\mathrm{BK}}(M(T_2), \overline{M}_1) \to \mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1)) \xrightarrow{\varpi} \mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))$$

Identifying $\mathrm{Hom}_{\mathrm{BK}}(M(T_2), \overline{M}_1) = \mathrm{Hom}_{\mathrm{BK}}(\overline{M}_2, \overline{M}_1)$ we see that the k_E-dimension of the ϖ-torsion subgroup of $\mathrm{Ext}^1_{\mathcal{O}}(M(T_2), M(T_1))$ equals

$$\dim_{k_E} \mathrm{Hom}_{\mathrm{BK}}(\overline{M}_2, \overline{M}_1) - \dim_{k_E} \mathrm{Hom}_{\mathrm{BK}}(M(T_2), M(T_1))/\varpi$$

By full faithfullness of $T \mapsto M(T)$ we have that $\mathrm{Hom}_{\mathrm{BK}}(M(T_2), M(T_1)) = \mathrm{Hom}_{\mathcal{O}[G_K]}(T_2, T_1)$. Since $\mathrm{Hom}_{\mathcal{O}[G_K]}(T_2, T_1)$ is $\mathcal{O}$-free and equals $\mathrm{Hom}_{E[G_K]}(V_2, V_1)$ after inverting p the lemma follows. □

2. Crystalline Liftings

As a consequence of Proposition 8.1.1 we can deduce a crystalline lifting result for Breuil–Kisin modules.

Definition 8.2.1. A crystalline $\mathcal{O}$-lattice T is obvious if it admits a composition series (i.e. a filtration by G_K-stable submodules

$$0 = T_n \subset \ldots \subset T_0 = T$$

such that each T_i/T_{i+1} is irreducible and p-torsion-free) such that for each i there is an unramified extension L_i/K and a crystalline character $\tilde{\zeta}_i : G_{L_i} \to \mathcal{O}^\times$ such that $T_i/T_{i+1} \cong \mathrm{Ind}_{L_i}^K \tilde{\zeta}_i$.

We remark that after [**2**, Lemma 1.4.3] obvious crystalline representations are potentially diagonalisable.

Theorem 8.2.2. *Let $M \in \mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ and suppose that $T(M)$ is cyclotomic-free (as in Definition 5.5.1). Suppose further that one of the following holds.*

1. *$T(M)$ is a successive extension of characters.*
2. $\mathrm{Weight}(M) \subset [0, p-1]$.
3. *$K = \mathbb{Q}_p$, every irreducible subquotient of $T(M)$ has k_E-dimension ≤ 4.*

Then there exists an obvious crystalline $\mathcal{O}$-lattice T with Hodge–Tate weights in $[0, p]$ such that $M(T)/\varpi = M$.

Proof. Argue by induction on the length of $T(M)$. If M has length one then the existence of a T follows in case (1) and (2) from Corollary 4.6.6 and Corollary 7.1.13 respectively. Case (3) follows from Lemma 7.2.9.

For the inductive step fit M into an exact sequence $0 \to N \to M \to P \to 0$ in $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$. Since $T(M)$ is cyclotomic-free N and P may be chosen such that $T(P)$ and $T(N)$ are as in Lemma 3.3.5, so that Proposition 3.3.4 applies to $\mathrm{Hom}(T(P), T(N))$. Our inductive hypothesis provides us with crystalline $\mathcal{O}$-lattices T_P and T_N with $M(T_P)/\varpi = P, M(T_N)/\varpi = N$. The result then follows immediately from Proposition 8.1.1. □

As an immediate consequence we deduce:

Corollary 8.2.3. *Let $\rho : G_K \to \mathrm{GL}(V)$ be a continuous representation of G_K on a finite dimensional k_E-vector space and assume that ρ is cyclotomic-free. For each $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, k_E)$ let W_τ be a multiset of integers in $[0, p]$ (not necessarily distinct). Consider the statement:*

ρ has a crystalline lift with τ-th Hodge–Tate weight W_τ if and only if ρ has an obvious crystalline lift with the same τ-th Hodge–Tate weights.

This statement is true in any of the following situations:

(1) *ρ is a successive extension of characters.*
(2) *$W_\tau \subset [0, p-1]$.*
(3) *$K = \mathbb{Q}_p$ and every irreducible subquotient of ρ has dimension ≤ 4.*

CHAPTER 9

Controlling the Shape of Crystalline Breuil–Kisin modules (after Gee–Liu–Savitt)

The aim of this chapter is to explain a small improvement in the work Gee–Liu–Savitt (Theorem 4.5.1). We hope this can be used to extend these methods to crystalline representations with Hodge–Tate weights outside the range $[0, p]$. The improvement involves proving Proposition 9.2.16 which was suggested to be true in [**16**, Remark 4.9].

We shall show that this at least allows us to extend results analogous to those above to crystalline representations with non-regular Hodge–Tate weights outside the range $[0, p]$. See Theorem 9.4.14 for a precise statement of the Hodge–Tate weights we consider.

It should be emphasised that most of the ideas in this chapter are ideas of Gee–Liu–Savitt. The only significant difference is our use of $B_{\max}$ in place of B_{crys} which allows for improvements in certain places, and our treatment of the case $p = 2$.

1. Period Rings Revisited

For this section we drop our assumption that K is unramified over $\mathbb{Q}_p$, thus K is a totally ramified extension of degree e over K_0 as described in the beginning of Chapter 2.

Recall from Construction 2.1.1 and Construction 2.1.2 where the rings $A_{\max}$ and $A_{\inf}$ are defined. Recall also the element $\mu = [\epsilon] - 1 \in A_{\inf}$, where $\epsilon \in \mathbb{Z}_p(1)$ is our fixed choice of $\mathbb{Z}_p$-generator.

Lemma 9.1.1. *For $n \geq 1$ we have $\frac{\varphi^{-1}(\mu)^{n-1}}{n} \in A_{\max}$.*

Proof. Observe that $\varphi^{-1}(\mu)^p \equiv \mu$ modulo $pA_{\inf}$ and so $\alpha = \frac{\varphi^{-1}(\mu)^p}{p} - \frac{\mu}{p} \in A_{\inf}$. The element $\varphi(\alpha)$ lies in the ideal

$$\{x \in A_{\inf} \mid \varphi^n(x) \in \ker\theta \text{ for all } n \geq 0\}$$

of $A_{\inf}$, which is known to be principal and generated by μ (see [**10**, Propostion 5.1.3]). Thus we may divide α by $\varphi^{-1}(\mu)$ and so deduce that $\frac{\varphi^{-1}(\mu)^{p-1}}{p} \in A_{\max}$. Write $n = p^s m$ with m coprime to p. As $p^s - 1 = (p-1)(1 + p + \ldots + p^{s-1})$ we have $n - 1 \geq p^s - 1 \geq (p-1)s$. Thus $\frac{\varphi^{-1}(\mu)^{n-1}}{p^s} \in A_{\max}$ and the lemma follows. $\square$

Lemma 9.1.2. *The element $\frac{t}{\mu} \in A_{\max}$ is a unit.*

Proof. Recall $t = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\mu^n}{n}$. Thus $-\frac{t}{\mu} = 1 + \mathcal{Q}$ where $\mathcal{Q} = \sum_{n\geq 2}\frac{(-\mu)^{n-1}}{n}$. If $n \geq 2$ then $\frac{(-\mu)^{n-1}}{n}$ is topologically nilpotent for the p-adic topology on $A_{\max}$. Thus $\mathcal{Q}$ is topologically nilpotent which implies the lemma. $\square$

For the next lemma we let $x \mapsto x(0)$ denote the map $A_{\mathrm{inf}} \to W(\overline{k})$ which lifts the projection $\mathcal{O}_{C^\flat} \to \overline{k}$ onto the residue field. Via the section of $\mathcal{O}_{C^\flat} \to \overline{k}$ which sends $y \in \overline{k}$ onto $([y], [y]^{1/p}, \dots) \in \mathcal{O}_{C^\flat}$ we view $x(0)$ as an element of A_{inf} for any $x \in A_{\mathrm{inf}}$. Note that $\theta(x(0)) = x(0)$.

Lemma 9.1.3. *Let $\nu \in A_{\mathrm{inf}}$ be a generator of $\ker\theta$. Then the infinite product $\lambda = \prod_{n\geq 0}\varphi^n(\frac{\nu}{\nu(0)})$ converges to an element of $A_{\max}$ with the following properties.*

(1) *$\varphi(\lambda)$ is a unit in $A_{\max}$*
(2) *$\frac{\lambda\varphi^{-1}(\mu)}{t} \in \frac{1}{p}A_{\max}$.*

Proof. We begin with two easy observations:

- If $x \in \mathcal{O}_{C^\flat}$ with $v^\flat(x) = 1$ then $\varphi^n([x]) \in p^{p^n}A_{\max}$. Indeed we can write $x^\sharp = py$ for some $y \in \mathcal{O}_C^\times$. Choose $y^\flat \in \mathcal{O}_{C^\flat}^\times$ with $(y^\flat)^\sharp = y$. Then $[x] - p[y^\flat]$ generates $\ker\theta$ and so $\frac{[x]-p[y^\flat]}{p} \in A_{\max}$, and $[x] \in pA_{\max}$.
- This also shows that if $x \in \mathcal{O}_{C^\flat}$ with $v^\flat(x) > 0$ then $[x]$ is topologically nilpotent in $A_{\max}$.

To show that the infinite product converges in $A_{\max}$ it suffices to show that $\varphi^n(\frac{\nu}{\nu(0)} - 1) \in A_{\max}$ and converges to zero p-adically. Since $\nu - \nu(0)$ lies in the kernel of $x \mapsto x(0)$ we can write

$$\nu - \nu(0) = \sum_{i\geq 0}[x_i]p^i$$

with $v^\flat(x_i) > 0$. We claim further that $v^\flat(x_0) = 1$. Since $\theta(\nu(0)) = \nu(0) = -x_0^\sharp - x_1^\sharp p - \dots$ and since every $v_p(x_i^\sharp) > 0$ we have $\nu(0) \equiv x_0^\sharp$ modulo $p^{1+\epsilon}\mathcal{O}_C$ where $\epsilon = \min\{v_p(x_1^\sharp), 1\}$. Hence our claim is equivalent to asking that $\nu(0) \in pW(\overline{k})^\times$. Let us show that this claim implies convergence of λ and (1) and (2). The claim and the first bullet point imply that $\frac{[x_0]}{\nu(0)} \in A_{\max}$ and so $\frac{\nu}{\nu(0)} - 1 \in A_{\max}$ can be written as

$$\frac{[x_0]}{\nu(0)} + \sum_{i\geq 1}[x_i]\alpha_i$$

with $v^\flat(x_i) > 0$ and $\alpha_i = \frac{p^i}{\nu(0)} \in W(\overline{k})$. The two bullet points above imply that $\varphi^n(\frac{\nu}{\nu(0)}-1)$ converges to zero in $A_{\max}$ so the product converges in $A_{\max}$. Moreover we see $\varphi^n(\frac{\nu}{\nu(0)}-1)$ is topologically nilpotent in $A_{\max}$ when $n \geq 1$ so $\varphi(\lambda)$ is a unit in $A_{\max}$. To prove (2) we need to show that $\lambda\varphi^{-1}(\mu) \in \frac{t}{p}A_{\max}$.

Lemma 9.1.2 shows that $\frac{t}{p}A_{\max} = \frac{\mu}{p}A_{\max}$. Thus (2) is implied by $\lambda \in \frac{\nu}{p}A_{\max}$ which follows because $\nu(0) \in pW(\bar{k})$.

It remains to prove the claim i.e. to prove that $v_p(\nu(0)) = 1$. Note that $[p^\flat] - p \in A_{\inf}$ is a generator of $\ker\theta$ if $p^\flat$ satisfies $(p^\flat)^\sharp = p$. Thus $\nu = \alpha([p^\flat] - p)$ for some unit $\alpha \in A_{\inf}$. We have $\nu(0) = -\alpha(0)p$ and since $\alpha(0) \in W(\bar{k})^\times$ this proves the claim. $\square$

LEMMA 9.1.4. *Let $a \in A_{\inf} \cap \mu^n B^+_{\max}$. Then $a \in \mu^n A_{\inf}$.*

PROOF. It suffices to treat the case $n = 1$. The map θ extends to a homomorphism $B^+_{\max} \to C$. In particular, for $m \geq 0$ we have $\varphi^m(a) \in \varphi^m(\mu)B^+_{\max}$ and so $\theta(\varphi^m(a)) = 0$. The lemma then follows again from the observation that the ideal

$$\{x \in A_{\inf} \mid \varphi^n(x) \in \ker\theta \text{ for all } n \geq 0\}$$

is generated by μ ([**10**, Proposition 5.1.3]). $\square$

As in Section 4 of Chapter 2 let π be a uniformiser of K and let $\pi^\flat \in \mathcal{O}_{C^\flat}$ be such that $(\pi^\flat)^\sharp = \pi$.

LEMMA 9.1.5. *If $a \in A_{\inf} \cap [\pi^\flat]^n B^+_{\max}$ then $a \in [\pi^\flat]^n A_{\inf}$.*

PROOF. In [**22**, Lemma 3.2.2] it is proved that $[\pi^\flat]^n B^+_{\mathrm{crys}} \cap A_{\inf} = [\pi^\flat]^n A_{\inf}$. Since $\varphi(B^+_{\max}) \subset B^+_{\mathrm{crys}} \subset B^+_{\max}$, if $b \in B^+_{\max}$ is such that $[\pi^\flat]^n b \in A_{\inf}$ then $[\pi^\flat]^{pn}\varphi(b) \in [\pi^\flat]^{pn}B^+_{\mathrm{crys}} \cap A_{\inf}$. Liu's result then implies $\varphi(b) \in A_{\inf}$ and so $b \in A_{\inf}$ also. $\square$

NOTATION 9.1.6. Recall $\mathcal{O}^{\mathrm{rig}}$ denotes the subring of $K_0[[u]]$ consisting of power series which converge on the open unit disk (Notation 2.4.7). Inside $\mathcal{O}^{\mathrm{rig}}$ we consider the subring $S_{\max} = W(k)[[u, \frac{u^e}{p}]] \cap \mathcal{O}^{\mathrm{rig}}$. Notice that for $f = \sum f_i u^i \in \mathcal{O}^{\mathrm{rig}}$ we have

$$f \in S_{\max} \Leftrightarrow v_p(f_i) + \lfloor \tfrac{i}{e} \rfloor \geq 0 \Leftrightarrow v_p(f_i) + \tfrac{i}{e} \geq 0$$

The second equivalence follows because $v_p(f_i) \in \mathbb{Z}$.

REMARK 9.1.7. We make two observations. The first is that $S_{\max}[\frac{1}{p}] = \mathcal{O}^{\mathrm{rig}}$. To see this recall $f = \sum f_i u^i \in \mathcal{O}^{\mathrm{rig}}$ if and only if $\sum f_i x^i$ converges for all $x \in C^\flat$ with $v_p(x) > 0$. In other words, if and only if $v_p(f_i) + ir \to \infty$ for every $r > 0$. In particular if $f \in \mathcal{O}^{\mathrm{rig}}$ then $v_p(f_i) + \frac{i}{e} \to \infty$ and so all but finitely many of the coefficients of f satisfy $v_p(f_i) + \frac{i}{e} \geq 0$. The second observation is that for any $f \in S_{\max}$ there are polynomials $q_i \in W(k)[u]$ of degree $< e$ converging p-adically to zero, such that

$$f = \sum (\tfrac{E}{p})^i q_i$$

where $E = E(u) \in W(k)[u]$ is the minimal polynomial of π over $W(k)$. To see this consider the following two subrings of $K_0[[u]]$.

$$R_1 = \{\sum (\tfrac{E}{p})^i q_i \mid q_i \to 0\}, \qquad S_1 = \{\sum (\tfrac{u^e}{p})^i q_i \mid q_i \to 0\}$$

In each case the $q_i \in W(k)[u]$ are polynomials of degree $< e$ which converge p-adically to zero (to see that R_1 is a ring one just has to check that any polynomial in $W(k)[u]$ lies in R_1 which can easily be checked by induction on the degree). It is clear both R_1 and S_1 are p-adically complete, and so since $\frac{u^e}{p} \in R_1$ and $\frac{E}{p} \in S_1$, we must have $R_1 = S_1$. Thus $S_{\max} = S_1 \cap \mathcal{O}^{\mathrm{rig}} = R_1 \cap \mathcal{O}^{\mathrm{rig}}$ and the claim follows.

LEMMA 9.1.8. *Let $f = \sum f_i u^i \in \mathcal{O}^{\mathrm{rig}}$. Then $f \in S_{\max}$ if and only if $\sum f_i[\pi^\flat]^i \in A_{\max}$.*

PROOF. From the remarks made in 9.1.7 we can write

$$f = \sum (\tfrac{E}{p})^i q_i$$

where the $q_i \in K_0[u]$ are polynomials of degree $< e$ converging to zero p-adically, and $f \in S_{\max}$ if and only if the q_i lie in $W(k)[u]$. We claim $\sum(\frac{E([\pi^\flat])}{p})^i q_i([\pi^\flat]) \in A_{\max}$ if and only if $q_i \in W(k)[u]$. To see this we use a result of Colmez. Following [**6**, §V.3] call an element x of B_{dR}^+ with $\theta(x) \neq 0$ flat if $x \in p^{w(x)} A_{\inf}$ where $w(x)$ is the integer part of $v_p(\theta(x))$. We also say 0 is flat. If $q_i = \sum_0^{e-1} a_i u^i$ is non-zero then $\theta(q_i([\pi^\flat])) = \sum_0^{e-1} a_i \pi^i$ is non-zero and, since $0 \leq v_p(\pi^i) < 1$ for all $0 \leq i < e$, the $v_p(a_i\pi^i)$ are distinct. Thus $w(q_i([\pi^\flat])) = \min v_p(a_i)$ and so $q_i([\pi^\flat]) \in p^{w(x)} A_{\inf}$. Therefore $q_i([\pi^\flat])$ is flat. Colmez shows in [**6**, Lemma V.3.1] that if $x \in B_{\mathrm{dR}}^+$ can be expressed as a sum $\sum_{n\geq 0} y_n(\frac{\nu}{p})^n$ with ν a generator of $\ker\theta \cap A_{\inf}$ and $y_n \in B_{\mathrm{dR}}^+$ flat, then $x \in A_{\max}$ if and only if $w(y_n) \geq 0$ and $w(y_n)$ converges to ∞. Applying this with $\nu = E([\pi^\flat])$ proves the lemma. □

2. Galois and Monodromy

We do not assume $K/\mathbb{Q}_p$ is unramified in this section.

NOTATION 9.2.1. Let T be a crystalline $\mathbb{Z}_p$-lattice, let $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and let $M(T)$ be the associated Breuil–Kisin module from Theorem 2.4.8. Then there are φ, G_{K_∞}-equivariant identifications

$$M(T) \otimes_{\mathfrak{S}} B_{\max} \cong T \otimes_{\mathbb{Z}_p} B_{\max} \cong D_{\mathrm{crys}}(V) \otimes_{K_0} B_{\max} \tag{9.2.2}$$

The middle and rightmost terms above admit natural G_K-actions which commute with φ. For these G_K-actions the right $\cong$ is G_K-equivariant. Via the above identification we obtain a G_K-action on $M(T) \otimes_{\mathfrak{S}} B_{\max}$.

LEMMA 9.2.3. *The G_K-action on* (9.2.2) *stabilises the submodule*

$$M(T)^\varphi \otimes_{\mathfrak{S}} B_{\max}^+ \cong D_{\mathrm{crys}}(V) \otimes_{K_0} B_{\max}^+$$

If we make $B_{\max}^+$ a topological ring by asserting that the $p^n A_{\max}$ form a system of open neighbourhoods of zero then this G_K-action is continuous.

PROOF. Clearly $D_{\mathrm{crys}}(V) \otimes_{K_0} B_{\max}^+$ is G_K-stable since $B_{\max}^+ \subset B_{\max}$ is G_K-stable. Since the action of G_K on $B_{\max}^+$ is also continuous the same is true for the action on $D_{\mathrm{crys}}(V) \otimes_{K_0} B_{\max}^+$. □

NOTATION 9.2.4. Let us write $\mathcal{M}(T) = M(T) \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{rig}}$. Recall $\lambda \in \mathcal{O}^{\mathrm{rig}}$ from Notation 2.4.7. From the sketched proof of Theorem 2.4.8 we have identifications

$$\mathcal{M}(T)[\tfrac{1}{\lambda}] \cong D_{\mathrm{crys}}(V) \otimes_{K_0} \mathcal{O}^{\mathrm{rig}}[\tfrac{1}{\lambda}] \tag{9.2.5}$$

as submodules of (9.2.2). We equip the modules in (9.2.5) with a differential operator $\mathcal{N}$ over $\partial = -u\frac{d}{du}$ by decreeing that $\mathcal{N}$ vanishes on $D_{\mathrm{crys}}(V)$. Since $\partial\varphi = p\varphi\partial$ we have the relation

$$\mathcal{N}\varphi = p\varphi\mathcal{N} \tag{9.2.6}$$

Additionally, since $\partial(\mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]) \subset u\mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$ we also have

$$\mathcal{N}(\mathcal{M}(T)[\tfrac{1}{\lambda}]) \subset u\mathcal{M}(T)[\tfrac{1}{\lambda}] \tag{9.2.7}$$

REMARK 9.2.8. The operator $\mathcal{N}$ is uniquely determined amongst those differential operators over ∂ on $\mathcal{M}(T)[\frac{1}{\lambda}]$ satisfying (9.2.6) and (9.2.7); indeed as φ is bijective on $D_{\mathrm{crys}}(V)$, if $\mathcal{N}$ is any such operator then for any $d \in D_{\mathrm{crys}}(V)$ can be written as $d = \varphi^n(d_n)$ for any $n \geq 1$. This implies $\mathcal{N}(d) \in \varphi^n(u\mathcal{M}(T)[\frac{1}{\lambda}])$ for all $n \geq 1$. As such $\mathcal{N}|_D = 0$.

LEMMA 9.2.9. *For all $m \in M(T)$ one has $\lambda\mathcal{N}(m) \in u\mathcal{M}(T)$.*

PROOF. With notation as in the sketched proof of Theorem 2.4.8 we have $\mathcal{M}(T) = \mathcal{M}(D)$ where $D = D_{\mathrm{crys}}(V)$. The construction of $\mathcal{M}(D)$ is given in [**20**, Subsection (1.2)]. In *loc. cit.* $\mathcal{M}(D)[\frac{1}{\lambda}] = D \otimes_{K_0} \mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$ is equipped with a differential operator N_∇ over $-u\lambda\frac{d}{du}$ by decreeing it vanishes on D. Thus $N_\nabla = \lambda\mathcal{N}$. In [**20**, Lemma 1.2.2] it is shown that N_∇ restricts to an operator $N_\nabla : \mathcal{M}(D) \to \mathcal{M}(D)$ which proves the lemma. □

NOTATION 9.2.10. For $\sigma \in G_K$ we let $\epsilon(\sigma)$ be the element of $\mathbb{Z}_p(1)$ defined by

$$\epsilon(\sigma)_n = \frac{\sigma(\pi^{1/p^n})}{\pi^{1/p^n}}$$

We remind the reader that the formal power series $\log([\epsilon(\sigma)])$ converges in B_{dR}^+ to αt where $\alpha \in \mathbb{Z}_p$ and t is the period defined in 2.1.1. In particular $\log([\epsilon(\sigma)]) \in A_{\max}$. If $\epsilon(\sigma)$ is a $\mathbb{Z}_p$-generator of $\mathbb{Z}_p(1)$ then $\alpha \in \mathbb{Z}_p^\times$.

LEMMA 9.2.11. *The action of G_K on $\mathcal{M}(T)[\frac{1}{\lambda}] \otimes_{\mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]} B_{\max}$ is given by*

$$\sigma(m \otimes a) = \sum_{n=0}^{\infty} \mathcal{N}^n(m) \otimes \sigma(a) \frac{(-\log([\epsilon(\sigma)]))^n}{n!}$$

PROOF. It is enough to show the above formula is valid on $\mathcal{M}(T)[\frac{1}{\lambda}]$. Using the identification (9.2.5) we view $D_{\mathrm{crys}}(V)$ as lying inside $\mathcal{M}(T)[\frac{1}{\lambda}]$. Any K_0-basis of $D_{\mathrm{crys}}(V)$ is an $\mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$-basis of $\mathcal{M}(T)[\frac{1}{\lambda}]$. Therefore it suffices to show the above formula is valid for $m = fd$ with $f \in \mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$ and $d \in D_{\mathrm{crys}}(V)$.

By definition $\mathcal{N}^n(fd) = \partial^n(f)d$. As the G_K-action on $D_{\mathrm{crys}}(V)$ is trivial we have $\sigma(fd) = \sigma(f)d$ where the action of G_K on f arises from the embedding of $\mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$ into $B_{\max}$. The lemma therefore reduces to the following computation: we must check that

$$\sum_{n=0}^{\infty} \frac{(-\log([\epsilon(\sigma)]))^n}{n!}\partial^n(f) \tag{9.2.12}$$

converges to $\sigma(f)$. It suffices to consider $f = [\pi^\flat]^i$. Then $\sigma(f) = [\epsilon(\sigma)]^i[\pi^\flat]^i$. On the other hand, since $\partial^n(f) = (-i)^n f$, the sum (9.2.12) equals $\exp(\log([\epsilon(\sigma)]^i))f$. If this sum converges then it will do so to $[\epsilon(\sigma)]^i f$ which proves the first part of the lemma. To show convergence it suffices to show that the sequence $\frac{(\log([\epsilon(\sigma)]))^n}{n!}$ lies in $A_{\max}$ and in this ring converges p-adically to zero. Since $\log([\epsilon(\sigma)] = \alpha t \in pA_{\max}$, if $\frac{p^n}{n!} \in \mathbb{Z}_p$ and tends p-adically to zero then will be done. Always $\frac{p^n}{n!} \in \mathbb{Z}_p$ and this sequence is a null-sequence when $p > 2$. However $\frac{2^n}{n!}$ is not 2-adically a null-sequence. To remedy the argument when $p = 2$ note that $\mu = ([\epsilon]^{1/2} + 1)([\epsilon]^{1/2} - 1)$ and so $\frac{\mu}{4} \in A_{\max}$. Lemma 9.1.2 then implies that $\log([\epsilon(\sigma)] \in 4A_{\max}$. As $\frac{4^n}{n!}$ is 2-adically a null-sequence everything is still valid. □

Lemma 9.2.13. *The module $M(T)^\varphi \otimes_{\mathfrak{S}} A_{\inf}$ is stable under the action of G_K. Moreover if $m \in \varphi(M(T))$ and $\sigma \in G_K$ then*

$$(\sigma - \mathrm{Id})^n(m) \in M(T)^\varphi \otimes_{\mathfrak{S}} [\pi^\flat]^p \mu^n A_{\inf}$$

when $n = 1$. If $\chi_{\mathrm{cyc}}(\sigma) = 1$ then the above is true for $n \geq 1$.

Proof. Lemma 9.2.3 tells us that $\sigma(m) \in M(T)^\varphi \otimes_{\mathfrak{S}} B^+_{\max}$ if $m \in M(T)^\varphi$. On the other hand, we know that as submodules of (9.2.2) we have

$$M(T)^\varphi \otimes_{\mathfrak{S}} A_{\inf}[\tfrac{1}{\mu}] = M(T) \otimes_{\mathfrak{S}} A_{\inf}[\tfrac{1}{\mu}] \cong T \otimes_{\mathbb{Z}_p} A_{\inf}[\tfrac{1}{\mu}]$$

Here we've used that $A_{\inf}[\frac{1}{\varphi^{-1}(\mu)}] \subset A_{\inf}[\frac{1}{\mu}]$ (as $\mu = \xi\varphi^{-1}(\mu)$). Since $A_{\inf}[\frac{1}{\mu}]$ is a G_K-stable subring of $B_{\max}$ it follows that $\sigma(m) \in M(T)^\varphi \otimes_{\mathfrak{S}} A_{\inf}[\frac{1}{\mu}]$. Lemma 9.1.4 implies that $B^+_{\max} \cap A_{\inf}[\frac{1}{\mu}] = A_{\inf}$ from which the first statement follows.

Now let us check the second statement. We first verify that $(\sigma - \mathrm{Id})^n(m) \in M(T)^\varphi \otimes_{\mathfrak{S}} [\pi^\flat]^p t^n B^+_{\max}$. We shall do this by checking that $(\sigma - \mathrm{Id})^n(m)$ may be expressed as

$$\sum_{j=n}^{\infty} \sum_{\substack{j_1+\ldots+j_n=j \\ j_i \geq 1}} \mathcal{N}^j(m) \otimes \frac{(-\log([\epsilon(\sigma)]))^j}{j_1!\ldots j_n!} \tag{9.2.14}$$

When $n = 1$ this is just Lemma 9.2.11. For $n > 1$ this will only be true if $\chi_{\mathrm{cyc}}(\sigma) = 1$. Arguing as in Lemma 9.2.11 one checks that (9.2.14) converges. If $\chi_{\mathrm{cyc}}(\sigma) = 1$ then $\sigma(\log[\epsilon(\sigma)]) = \log([\epsilon(\sigma)])$. Using Lemma 9.2.11 and the continuity of the action of σ it is easy to check that

$$(\sigma-\mathrm{Id})\left(\sum_{j=n}^{\infty}\sum_{\substack{j_1+\ldots+j_n=j\\ j_i\geq 1}}\mathcal{N}^j(m)\otimes\frac{(-n\log([\epsilon(\sigma)]))^j}{j_1!\ldots j_n!}\right)=\sum_{i=n+1}^{\infty}\sum_{\substack{i_1+\ldots+i_{n+1}=i\\ i_l\geq 1}}\mathcal{N}^i(m)\otimes\frac{(-\log([\epsilon(\sigma)]))^i}{i_1!\ldots i_{n+1}!}$$

Inducting on n we see that (9.2.14) equals $(\sigma-\mathrm{Id})^n(m)$. Iterating the relation $p\varphi\mathcal{N}=\mathcal{N}\varphi$ gives $p^j\varphi\mathcal{N}^j=\mathcal{N}^j\varphi$. If $m=\varphi(m')$ then $\mathcal{N}^j(m')\in M(T)\otimes_{\mathfrak{S}} u\mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}]$ and so $\mathcal{N}^j(m)\in M(T)^{\varphi}\otimes_{\mathfrak{S}} u^p\mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$. Therefore each term of (9.2.14) lies in $M(T)^{\varphi}\otimes_{\mathfrak{S}}[\pi^\flat]^p t^n B^+_{\max}$. Lemma 2.1.3 implies $[\pi^\flat]^p t^n B^+_{\max}$ is closed inside $B^+_{\max}$, and as a consequence $M(T)^{\varphi}\otimes_{\mathfrak{S}}[\pi^\flat]^p t^n B^+_{\max}$ is closed in $M(T)^{\varphi}\otimes_{\mathfrak{S}} B^+_{\max}$. Thus (9.2.14) is in $M(T)^{\varphi}\otimes_{\mathfrak{S}}[\pi^\flat]^p t^n B^+_{\max}$ which proves our claim.

This claim, the statement of the first part of the lemma, and the fact that $t^n[\pi^\flat]^p B^+_{\max}\cap A_{\mathrm{inf}}=\mu^n[\pi^\flat]^p A_{\mathrm{inf}}$ together imply the lemma. To check the last fact recall that $\frac{t}{\mu}$ is a unit in $A_{\max}$ and so $t^n[\pi^\flat]^p B^+_{\max}=\mu^n[\pi^\flat]^p B^+_{\max}$. Now use Lemma 9.1.4 and Lemma 9.1.5. □

Lemma 9.2.15. *If $\sigma\in G_K$ is such that $\chi_{\mathrm{cyc}}(\sigma)=1$ then the operator $\mathcal{N}$ on $\mathcal{M}(T)[\frac{1}{\lambda}]$ is given by*

$$\mathcal{N}(m)=\frac{-1}{\log([\epsilon(\sigma)])}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(\sigma-\mathrm{Id})^n}{n}(m)$$

Proof. As in the proof of Lemma 9.2.11 it suffices to show the formula is valid for $m=fd$ with $d\in D_{\mathrm{crys}}(V)$ and $f\in\mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$. Note that $\mathcal{N}(m)=\partial(f)d$ and $(\sigma-\mathrm{Id})^n(m)=(\sigma-\mathrm{Id})^n(f)d$. Thus checking the validity of the formula amounts to checking that for $f\in\mathcal{O}^{\mathrm{rig}}[\frac{1}{\lambda}]$ the following sum converges to $\partial(f)$.

$$\frac{-1}{\log([\epsilon(\sigma)])}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(\sigma-\mathrm{Id})^n}{n}(f)$$

We may assume that $f=u^i=[\pi^\flat]^i$. Since $\chi_{\mathrm{cyc}}(\sigma)=1$ we have $\sigma([\epsilon(\sigma)])=[\epsilon(\sigma)]$, so $(\sigma-\mathrm{Id})^n([\pi^\flat]^i)=([\epsilon(\sigma)]^i-1)^n[\pi^\flat]^i$. We remark that $[\epsilon(\sigma)]^i-1$ is divisible by μ in A_{inf} and therefore $\frac{([\epsilon(\sigma)]^i-1)^n}{n}$ converges p-adically to zero in $A_{\max}$. As such the infinite sum above converges to $\frac{-\log([\epsilon(\sigma)]^i)}{\log(\epsilon[\sigma])}[\pi^\flat]^i=-i[\pi^\flat]^i=\partial(f)$. □

Recall $S_{\max}=W(k)[[u,\frac{u^e}{p}]]\cap\mathcal{O}^{\mathrm{rig}}$. We equip this ring with the φ it inherits from the φ on $\mathcal{O}^{\mathrm{rig}}$. After Lemma 9.1.8 we have $S_{\max}=A_{\max}\cap\mathcal{O}^{\mathrm{rig}}$.

Proposition 9.2.16. *Let $m\in M(T)$. Suppose that $K_\infty\cap K(\mu_{p^\infty})=K$. Then $\lambda\mathcal{N}(m)\in M(T)\otimes_{\mathfrak{S}}\frac{u}{p}S_{\max}$.*

Proof. The assumption that $K_\infty\cap K(\mu_{p^\infty})=K$ implies that there exists $\sigma\in G_K$ such that $\chi_{\mathrm{cyc}}(\sigma)=1$ and such that $\log([\epsilon(\sigma)])=\alpha t$ for some $\alpha\in\mathbb{Z}_p^\times$.

Observe how the second statement of Lemma 9.2.13 is equivalent to asking that $(\sigma - \mathrm{Id})^n(m) \in M(T) \otimes \varphi^{-1}(\mu)^n[\pi^\flat]A_{\mathrm{inf}}$. By Lemma 9.1.1 $\varphi^{-1}(\mu^n)[\pi^\flat]A_{\mathrm{inf}} \subset \varphi^{-1}(\mu)[\pi^\flat]A_{\max}$. Thus each term of

$$\mathcal{N}(m) = \frac{-1}{\log([\epsilon(\sigma)])}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(\sigma - \mathrm{Id})^n}{n}(m)$$

lies in $M(T) \otimes_{\mathfrak{S}} \frac{[\pi^\flat]\varphi^{-1}(\mu)}{t}A_{\max}$. The same is true for the entire sum which follows because, arguing as in Lemma 2.1.3,

$$\tfrac{[\pi^\flat]\varphi^{-1}(\mu)}{t}A_{\max} \subset A_{\max}$$

is closed for the p-adic topology. Applying Lemma 9.1.3 with $\nu = E([\pi^\flat])$ gives $\frac{\lambda\varphi^{-1}(\mu)}{t} \in \frac{1}{p}A_{\max}$. We then deduce that $\lambda\mathcal{N}(m) \in M(T) \otimes_{\mathfrak{S}} \frac{[\pi^\flat]}{p}A_{\max}$. After Lemma 9.2.9 we know also that $\lambda\mathcal{N}(m) \in M(T)\otimes_{\mathfrak{S}} u\mathcal{O}^{\mathrm{rig}}$. The lemma now follows because $\frac{[\pi^\flat]}{p}A_{\max} \cap u\mathcal{O}^{\mathrm{rig}} = \frac{u}{p}S_{\max}$ (see Lemma 9.1.8). ☐

Remark 9.2.17. Recall Remark 4.5.2 with regards the condition $K_\infty \cap K(\mu_{p^\infty}) = K$.

Remark 9.2.18. This result provides an improvement of [**16**, Proposition 4.7] as suggested might be possible in [**16**, Remark 4.9].

3. Filtrations on Crystalline Breuil–Kisin Modules

Throughout this section, unless otherwise stated, T denotes a crystalline $\mathbb{Z}_p$-lattice, $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and $M = M(T)$. Additionally let us write D in place of $D_{\mathrm{crys}}(V)$.

Lemma 9.3.1. *Make $M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}}$ into an object of $\mathrm{Fil}(\widehat{\mathfrak{S}})$ by setting $F^i(M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}}) = (M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}}) \cap E^i(M \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}})$. Then there is a natural exact sequence of $\widehat{\mathfrak{S}}$-modules*

$$0 \to M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}} \xrightarrow{E} M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}} \xrightarrow{f_\pi} D_K \to 0$$

Moreover, the map f_π is a strict morphism in $\mathrm{Fil}(\widehat{\mathfrak{S}})$.

Proof. This follows from Theorem 2.4.8. The second identification of this theorem tells us that $M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}} = D_K \otimes_K \widehat{\mathfrak{S}}$. Thus we obtain the map f_π by reducing this identification modulo E. To see that f_π is strictly compatible with the filtrations use the final identification of Theorem 2.4.8, which tells us that $M \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}} = F^0(D_K \otimes_K \widehat{\mathfrak{S}}[\frac{1}{E}])$. We therefore have

$$F^i(M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}}) = E^i F^0\big(D_K \otimes_K \widehat{\mathfrak{S}}[\tfrac{1}{E}]\big) \cap \big(D_K \otimes_K \widehat{\mathfrak{S}}\big) = \sum F^j D_K \otimes_K E^{\max\{0,i-j\}}\widehat{\mathfrak{S}}$$

and so $f_\pi(F^i(M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}})) = \sum_{i \leq j} F^j D_K = F^i D_K$. ☐

Notation 9.3.2. Let $\underline{\mathcal{M}} = M \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}]$. As usual write $\underline{\mathcal{M}}^\varphi$ for the $\mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}]$-module generated by $\varphi(\underline{\mathcal{M}})$. Equivalently $\underline{\mathcal{M}}^\varphi = M^\varphi \otimes_{\mathfrak{S}} \mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}]$.

LEMMA 9.3.3. *Equip M^φ and $\underline{\mathcal{M}}^\varphi$ respectively with filtrations $F^iM^\varphi = M^\varphi \cap E^iM$ and $F^i\underline{\mathcal{M}}^\varphi = \underline{\mathcal{M}}^\varphi \cap E^i\underline{\mathcal{M}}$.*

(1) *The inclusions $M^\varphi \to \underline{\mathcal{M}}^\varphi \to M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}}$ are strict morphisms in* $\mathrm{Fil}(\mathfrak{S})$.
(2) *The restriction of f_π induces an exact sequence of $\mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}]$-modules*

$$0 \to \underline{\mathcal{M}}^\varphi \xrightarrow{E} \underline{\mathcal{M}}^\varphi \xrightarrow{f_\pi} D_K \to 0$$

and f_π is strict as a morphism of filtered modules.
(3) *The restriction of f_π to M^φ induces an exact sequence of $\mathfrak{S}$-modules*

$$0 \to M^\varphi \xrightarrow{E} M^\varphi \to M_{f_\pi} \to 0$$

where $M_{f_\pi} := f_\pi(M^\varphi) \subset D_K$. If M_{f_π} is given the subspace filtration coming from D_K then f_π becomes strict after inverting p.

PROOF. Since all the modules in the lemma are free over the respective rings each of (1),(2) and (3) follow from the observation that for $j \geq 1$ the inclusions $\mathfrak{S} \to \mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}] \to \widehat{\mathfrak{S}}$ induce maps

$$\mathfrak{S}/E^j \hookrightarrow \mathcal{O}^{\mathrm{rig}}[\tfrac{1}{\varphi(\lambda)}]/E^j \xrightarrow{\sim} \widehat{\mathfrak{S}}/E^j$$

in which the first inclusion becomes an isomorphism after inverting p. Let us explain this only for (3), the other parts follow by similar arguments. Since $\mathfrak{S}/E \to \widehat{\mathfrak{S}}/E$ is injective it follows that $M^\varphi \cap (EM^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}}) = EM^\varphi$; hence the kernel of $f_\pi : M^\varphi \to D_K$ is EM^φ. This shows the sequence in (3) is exact. To show it becomes strict after inverting p take $i \in \mathbb{Z}$; we must show $f_\pi(F^iM^\varphi)[\frac{1}{p}] = f_\pi(F^i(M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}}))$. Let $j \geq 1$ be large enough that $E^j(M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}}) \subset F^i(M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}})$. As $\mathfrak{S}/E^j \to \widehat{\mathfrak{S}}/E^j$ is surjective after inverting p we can write any $\widehat{m} \in F^i(M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}})$ as $m + E^j\widehat{n}$ for some $m \in M^\varphi[\frac{1}{p}]$, $\widehat{n} \in M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}}$. We see $m \in E^i(M \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}})$ and so by (1) $m \in F^i(M^\varphi)[\frac{1}{p}]$. Thus

$$(F^iM^\varphi)[\tfrac{1}{p}] + E^j(M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}}) = F^i(M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}})$$

and that $f_\pi(F^iM^\varphi[\frac{1}{p}]) = f_\pi(F^i(M^\varphi \otimes_{\mathfrak{S}} \widehat{\mathfrak{S}}))$ follows.

It remains to prove the claim concerning $\mathfrak{S}/E^j \to \mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}]/E^j \to \widehat{\mathfrak{S}}/E^j$. The injectivity of these maps follows because for $A \in \{\mathfrak{S}, \mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}]\}$, $E^i\widehat{\mathfrak{S}} \cap A[\frac{1}{E}] = E^iA$. Surjectivity follows because $\widehat{\mathfrak{S}}$ is the E-adic completion of $\mathfrak{S}[\frac{1}{p}]$ so $\widehat{\mathfrak{S}}/E^j = \mathfrak{S}[\frac{1}{p}]/E^j$. $\square$

REMARK 9.3.4. The map $f_\pi : M^\varphi \to M_{f_\pi}$ will not in general be strict. As we discuss below, when f_π is strict we obtain comparison between the weights of M/p (in the sense of Definition 4.3.3) and the Hodge–Tate weights of T.

LEMMA 9.3.5. *For any $\mathfrak{S}$-basis $\underline{e}$ of M the determinant of the matrix $X \in \mathrm{GL}_n(\mathfrak{S}[\frac{1}{E}])$ is defined by $\varphi(\underline{e}) = \underline{e}X$ has E-adic valuation*

$$\sum_{i \in \mathrm{HT}(V)} i$$

(i.e. is this power of E multiplied by a unit in $\mathfrak{S}$).

PROOF. The E-adic valuation of X does not change if we replace $\underline{e}$ by a basis of $\underline{\mathcal{M}}$ so we are free to take $\underline{e}$ to be any $\mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}]$-basis of $\underline{\mathcal{M}}$. The map $f_\pi : \underline{\mathcal{M}}^\varphi \to D_K$ is strict and so choosing K-bases of $\mathrm{gr}^i(D_K)$ and applying Lemma 4.1.8 applied with $A = \mathcal{O}^{\mathrm{rig}}$, $a = E$ and $M = \underline{\mathcal{M}}$, $N = \underline{\mathcal{M}}^\varphi$, we obtain a basis $\underline{e}$ of $\underline{\mathcal{M}}$ such that $\underline{e}\,\mathrm{diag}(E^{r_i})$ is a basis of $\underline{\mathcal{M}}^\varphi$ where $\mathrm{HT}(V) = \{r_i\}$. Thus $\varphi(\underline{e}) = \underline{e}\,\mathrm{diag}(E^{r_i})A$ for some $A \in \mathrm{GL}_n(\mathcal{O}^{\mathrm{rig}}[\frac{1}{\varphi(\lambda)}])$ which proves the lemma. $\square$

LEMMA 9.3.6. *Suppose K is unramified over $\mathbb{Q}_p$. Let $\overline{M} = M/p$ which is an object of $\mathrm{Mod}_k^{\mathrm{BK}}$. Suppose that one of either $f_\pi : M^\varphi \to M_{f_\pi}$ or $q : M^\varphi \to \overline{M}^\varphi$ is strict. Then there are isomorphisms of filtered modules $\overline{M}^\varphi/u \to M_{f_\pi}/p$, functorial in T.*

PROOF. As $K = K_0$ is unramified over $\mathbb{Q}_p$ we have that $E = u - \pi$. Thus as k-vector spaces we can identify $\overline{M}^\varphi/u$ and M_{f_π}/p with $M^\varphi/(u,p) =: \overline{M}_k^\varphi$ functorially in T. Thus we have a commutative diagram

$$\begin{array}{ccc} M^\varphi & \xrightarrow{f_\pi} & M_{f_\pi} \\ \downarrow{\scriptstyle q} & & \downarrow \\ \overline{M}^\varphi & \longrightarrow & \overline{M}_k^\varphi \end{array}$$

We can equip $\overline{M}_k^\varphi$ with two filtrations: the quotient filtration $F^i\overline{M}_k^\varphi$ coming from the surjection $\overline{M}^\varphi \to \overline{M}_k^\varphi$ (this is the filtration as defined in Lemma 4.3.2), and the quotient filtration coming from the surjection $M_{f_\pi} \to \overline{M}_k^\varphi$ (where M_{f_π} is equipped with the subspace filtration coming from its inclusion into D_K) which we shall denote by $G^i\overline{M}_k^\varphi$. It suffices to prove that if either f_π or q are strict then $G^i\overline{M}_k^\varphi = F^i\overline{M}_k^\varphi$.

If f_π is strict then $G^i \subset F^i$ and if q is strict then $F^i \subset G^i$. Thus after Corollary 4.1.13 it suffices to show that

$$\sum i \dim_k \mathrm{gr}^i_F(\overline{M}_k^\varphi) = \sum i \dim_k \mathrm{gr}^i_G(\overline{M}_k^\varphi)$$

We prove this equality by showing both the left and right hand sides are equal to $\sum_{i \in \mathrm{HT}(V)} i$. First consider the right hand side: since $M_{f_\pi} \subset D_K$ is strict $\mathrm{gr}^i(M_{f_\pi}) \subset \mathrm{gr}^i(D_K)$ with p-torsion cokernel. Thus $\sum_{i \in \mathrm{HT}(V)} i = \sum i \dim_k \mathrm{gr}^i(M_{f_\pi})/p$. On the other hand, since $M_{f_\pi} \to \overline{M}_k^\varphi$ is strict when $\overline{M}_k^\varphi$ is given the filtration G^i, it follows that $\mathrm{gr}^i_G(\overline{M}_k^\varphi) = \mathrm{gr}^i(M_{f_\pi})/\mathrm{gr}^i(pM_{f_\pi})$. Since $F^i(pM_{f_\pi}) = pF^i(M_{f_\pi})$ we have $\sum i \dim_k \mathrm{gr}^i_G(\overline{M}_k^\varphi) = \sum_{i \in \mathrm{HT}(V)} i$. For the left hand side: by definition $\sum i \dim_k \mathrm{gr}^i_F(\overline{M}_k^\varphi) = \sum_{i\, \mathrm{Weight}(\overline{M})} i$. By

Remark 4.3.6 this sum is just the u-adic valuation of the determinant of the matrix of $\varphi : \overline{M} \to \overline{M}[\frac{1}{u}]$ in any basis. This is equal to the E-adic valuation of the determinant of $\varphi : M \to M[\frac{1}{E}]$ in any basis as in Lemma 9.3.5 and this lemma shows $\sum_{i \in \mathrm{HT}(V)} i$. □

Corollary 9.3.7. *Assume $K = K_0$ and that T is a crystalline $\mathcal{O}$-lattice. If f_π is strict then* $\mathrm{Weight}_\tau(M/\varpi) = \mathrm{HT}_\tau(V)$.

Proof. Functoriallity of the identification of Lemma 9.3.6 implies the identification $\overline{M}^\varphi/u = M_{f_\pi}/p$ is $\mathcal{O}$-linear. Thus $M^\varphi/(\varpi, u) = M_{f_\pi}/\varpi$ in $\mathrm{Fil}(k_E)$. Arguing as in the proof of Lemma 9.3.6, the multiset containing i with multiplicity equal to $\dim_{k_E} \mathrm{gr}^i(M_{f_\pi,\tau}/\varpi)$ is the multiset $\mathrm{HT}_\tau(V)$. From this the result follows. □

4. Strictness of f_π

Keep the notation of the previous section, so that (unless otherwise stated) T denotes a crystalline $\mathbb{Z}_p$-lattice, $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and $M = M(T)$.

Lemma 9.4.1. *The operator $\mathcal{N}$ sends $F^i \underline{\mathcal{M}}^\varphi$ onto $F^{i-1}\underline{\mathcal{M}}^\varphi$.*

Proof. If $m \in \underline{\mathcal{M}}^\varphi$ then, using that $\mathcal{N}\varphi = p\varphi\mathcal{N}$ and Lemma 9.2.9, it follows that $\mathcal{N}(m) \in \underline{\mathcal{M}}^\varphi$. If $m \in E^i \underline{\mathcal{M}}$, using the Leibnitz rule, we see also that $\mathcal{N}(m) \in E^{i-1}\underline{\mathcal{M}}$. □

Lemma 9.4.2. *A necessary and sufficient condition that $x \in \underline{\mathcal{M}}^\varphi$ lie in $F^i\underline{\mathcal{M}}^\varphi$ is that $f_\pi(x) \in F^i D_K$ and $\mathcal{N}(x) \in F^{i-1}\underline{\mathcal{M}}^\varphi$.*

Proof. Our proof is based upon the observations made in [**17**, Proposition 2.1.12]. Temporarily write $F^i = F^i\underline{\mathcal{M}}^\varphi$. If we define

$$\widetilde{F}^i = \{x \in \underline{\mathcal{M}}^\varphi \mid N(x) \in F^{i-1}, f_\pi(x) \in F^i D_K\}$$

then we must show $F^i = \widetilde{F}^i$ for each i. Lemma 9.4.1 and the fact that $f_\pi(F^i) = F^i D_K$ tells us that $F^i \subset \widetilde{F}^i$. For small enough i we have $F^i = \underline{\mathcal{M}}^\varphi$ and so, for such i, $F^i = \widetilde{F}^i$ is immediate. Using this observation as the base case we prove equality for all i by induction.

Thus we suppose that $F^j = \widetilde{F}^j$ for $j < i$. If $x \in \widetilde{F}^i$ then, since $f_\pi(F^i) = F^i D_K$, we can write $x = y + Ez$ with $y \in F^i$ and $z \in \underline{\mathcal{M}}^\varphi$. We see that $Ez \in \widetilde{F}^i$ and we claim this implies $z \in \widetilde{F}^{i-1} = F^{i-1}$. If this were the case then we would have $Ez \in F^i$ and so $x \in F^i$.

Since $Ez \in \widetilde{F}^i$ one has $\mathcal{N}(Ez) = E\mathcal{N}(z) + \partial(E)z \in F^{i-1}$. As $\partial(E)$ is not divisible by E it follows that $f_\pi(z) \in F^{i-1}D_K$. In fact, for $l \geq 0$ we have

$$\mathcal{N}^{l+1}(Ez) = \sum_{j=0}^{l+1} \binom{l+1}{j} \partial^j(E)\mathcal{N}^{l+1-j}(z)$$

and so by induction on l it follows that $f_\pi(\mathcal{N}^l(z)) \in F^{i-1-l}D_K$ for $l \geq 0$. As such in order to show that $z \in F^{i-1}$ we just need to show $\mathcal{N}(z) \in F^{i-2}$. We shall prove more, namely that $\mathcal{N}^l(z) \in F^{i-1-l}$ for $l > 0$. For large enough l

one knows that $\mathcal{N}^l(z) \in F^{i-1-l} = \underline{\mathcal{M}}^\varphi$. Then we argue by decreasing induction on l: from the above we know $f_\pi(\mathcal{N}^l(z)) \in F^{i-1-l}D_K$ and, by inductive hypothesis, $\mathcal{N}(\mathcal{N}^l(z)) \in F^{i-2-l}$. We conclude that $\mathcal{N}^l(z) \in \widetilde{F}^{i-1-l} = F^{i-1-l}$ which completes the proof. $\square$

HYPOTHESIS 9.4.3. From now on we assume K is unramified over $\mathbb{Q}_p$. Thus $K = K_0$ and $E(u) = u - \pi$.

DEFINITION 9.4.4. Let $H(u) = \frac{u-\pi}{\pi}$. For $x \in \underline{\mathcal{M}}^\varphi$ inductively define $x^{(i)}$ by $x^{(0)} = x$ and

$$x^{(i)} = \sum_{l=0}^{i-1} \frac{H(u)^l}{l!} \mathcal{N}^l(x^{(i-1)}) \in \underline{\mathcal{M}}^\varphi$$

LEMMA 9.4.5. *Suppose that* $F^0\underline{\mathcal{M}}^\varphi = \underline{\mathcal{M}}^\varphi$*. Then for all* $x \in \underline{\mathcal{M}}^\varphi$ *with* $f_\pi(x) \in F^r D_K$ *we have* $x^{(i)} \in F^{\delta_i}\underline{\mathcal{M}}^\varphi$ *where* $\delta_i = \min\{i, r\}$*.*

PROOF. We argue by induction on i. The claim when $i = 0$ is obvious. For general i, using Lemma 9.4.2, it suffices to show that $\mathcal{N}(x^{(i)}) \in F^{\delta_i - 1}\underline{\mathcal{M}}^\varphi$. Since $\delta_{i-1} \geq \delta_i - 1$ it is actually enough to show $\mathcal{N}(x^{(i)}) \in F^{\delta_{i-1}}\underline{\mathcal{M}}^\varphi$. We compute that

$$\begin{aligned}\mathcal{N}(x^{(i)}) &= \sum_{l=0}^{i-1} \Big(\frac{H(u)^{l-1}\partial(H(u))}{(l-1)!} \mathcal{N}^l(x^{(i-1)}) + \frac{H(u)^l}{l!} \mathcal{N}^{l+1}(x^{(i-1)}) \Big) \\ &= \underbrace{\frac{H(u)^{i-1}}{(i-1)!} \mathcal{N}^i(x^{(i-1)})}_{(a)} + \sum_{l=1}^{i-1} \underbrace{(1 + \partial H(u)) \frac{H(u)^{l-1}}{(l-1)!} \mathcal{N}^l(x^{(i-1)})}_{(b)}\end{aligned}$$

Since $F^0\underline{\mathcal{M}}^\varphi = \underline{\mathcal{M}}^\varphi$ the term (a) above lies in $F^{i-1}\underline{\mathcal{M}}^\varphi \subset F^{\delta_{i-1}}\underline{\mathcal{M}}^\varphi$. From Lemma 9.4.1 and the inductive hypothesis that $x^{(i-1)} \in F^{\delta_{i-1}}\underline{\mathcal{M}}^\varphi$ we deduce that $\mathcal{N}^l(x^{(i-1)}) \in F^{\delta_{i-1}-l}\underline{\mathcal{M}}^\varphi$. Then, since $1 + \partial H(u) = -H(u)$, we deduce that each summand of (b) also lies in $F^{\delta_{i-1}}\underline{\mathcal{M}}^\varphi$. $\square$

LEMMA 9.4.6. *Let* $n \geq 0$ *and let* $f \in \varphi^n(\frac{u}{p}W(k)[[\frac{u}{p}]])$*. If* f *converges around* $u = \pi$ *then* $f = \sum_0^\infty \alpha_i(u-\pi)^i$ *for some* $\alpha_i \in K_0$ *such that if* $0 \leq m \leq p^n$ *then* $v_p(\alpha_m) \geq p^n - 1 - m + v_p\binom{p^n}{m}$*.*

PROOF. There are $g_j \in W(k)$ such that $f = \sum_{j=1}^\infty \frac{g_j}{\pi^j} u^{p^n j}$. On the other hand since f converges around $u = \pi$ it has a Taylor expansion $\sum_{i=0}^\infty \alpha_i(u-\pi)^i$ for some $\alpha_i \in K_0$. For any $m \geq 0$ we therefore have

$$\alpha_m = \tfrac{1}{m!}(\tfrac{d}{du})^m(f)|_{u=\pi}$$

and so

$$\alpha_m = \sum_{j=1}^\infty \binom{p^n j}{m} g_j \pi^{(p^n-1)j-m}$$

To prove the lemma it suffices to show that for $j \geq 1$ and $0 \leq m \leq p^n$

$$v_p\binom{p^n j}{m} + (p^n - 1)j \geq v_p\binom{p^n}{m} + p^n - 1$$

Since $\binom{p^n j}{m} = \frac{1}{m!}(p^n j)(p^n j - 1)\dots(p^n j - m + 1)$ and $v_p(p^n j - i) = v_p(i)$ for $0 < i < m$ we have $v_p\binom{p^n j}{m} = v_p(p^n j) + v_p((m-1)!) - v_p(m!) = v_p(p^n j) - v_p(m)$ if $m \geq 1$ and 0 if $m = 0$. This proves the inequality and therefore the lemma. $\square$

LEMMA 9.4.7. *Let $a \geq 1$ and $k \geq 0$. Then $\frac{\partial^k(H(u)^a)}{a!}$ is a $\mathbb{Z}$-linear combination of $\frac{H(u)^{a'}}{a'!}$ for $1 \leq a' \leq a$.*

PROOF. If $k = 0$ there is nothing to prove. Since $\partial(H(u)) = -1 - H(u)$ we have that $\partial(H(u)^a) = aH(u)^{a-1}(-1 - H(u))$ and so $\partial^k(H(u)^a)/a!$ equals

$$\begin{aligned}\frac{1}{a!}\partial^{k-1}(aH(u)^{a-1}(-1-H(u))) &= \frac{1}{(a-1)!}\sum_{j=0}^{k-1}\binom{k-1}{j}\partial^j(H(u)^{a-1})\partial^{k-1-j}(-1-H(u)) \\ &= \frac{1}{(a-1)!}(-1-H(u))\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^{k-1-j}\partial^j(H(u)^{a-1})\end{aligned}$$

if $k > 0$ (for the second equality we use that $\partial^n(-1 - H(u)) = (-1)^n(-1 - H(u))$). The lemma follows easily by induction on k. $\square$

LEMMA 9.4.8. *For any $x \in \underline{\mathcal{M}}^\varphi$ and any $i \geq 0$ the elements $x^{(i)} - x$ may be written as a $\mathbb{Z}$-linear combination of terms of the form*

$$\frac{H(u)^a}{a!}\mathcal{N}^b(x)$$

with $a, b \geq 1$.

PROOF. We argue by induction on i. The claim is clearly true when $i = 0$ and $i = 1$; thus assume the lemma holds for $x^{(i-1)} - x$. We can write

$$x^{(i)} - x = x^{(i-1)} - x + \sum_{l=1}^{i-1}\frac{H(u)^l}{l!}\mathcal{N}^l(x^{(i-1)})$$

Therefore our inductive hypothesis tells us that $x^{(i)} - x$ is a $\mathbb{Z}$-linear combination of the terms of the form $\frac{H(u)^l}{l!}\mathcal{N}^l(\frac{H(u)^a}{a!}\mathcal{N}^b(x))$ with $a, b \geq 1$ and $l \geq 0$. We need to show that when $l \geq 1$ these terms may be expressed as in the lemma. We can write

$$\frac{H(u)^l}{l!}\mathcal{N}^l\left(\frac{H(u)^a\mathcal{N}^b(x)}{a!}\right) = \sum_{k=0}^{l}\binom{l}{k}\frac{H(u)^l\partial^k(H(u)^a)}{a!l!}\mathcal{N}^{l-k+b}(x) \tag{9.4.9}$$

Using the previous lemma we deduce that (9.4.9) may be expressed as an $\mathbb{Z}$-linear combination of terms of the form

$$\frac{H(u)^{l+a'}}{a'!l!}\mathcal{N}^{l-k+b}(x)$$

with $0 \le a' \le a$. Using that $x!y!$ divides $(x+y)!$ we deduce the lemma. $\square$

LEMMA 9.4.10. *Suppose that $F^0\underline{\mathcal{M}}^\varphi = \underline{\mathcal{M}}^\varphi$ and $K_\infty \cap K(\mu_{p^\infty}) = K$. Let $b, n \ge 1$ and $e_1, \dots, e_d$ be a basis of M^φ. Suppose that $x \in \varphi^n(M)$. Then we can write*

$$\mathcal{N}^b(e_j) = \sum_{k=0}^{d} \sum_{m=0}^{\infty} \beta_{k,m}(u-\pi)^m e_k$$

such that for $0 \le m \le p^n$ one has $v_p(\beta_{k,m}) \ge nb + p^n - m - 1 + v_p\binom{p^n}{m}$.

PROOF. Suppose $x = \varphi^n(y)$ for some $y \in M$. Proposition 9.2.16 says that $\mathcal{N}(y) \in M \otimes_{\mathfrak{S}} \frac{u}{p} S_{\max}[\frac{1}{\lambda}]$. The fact that $\partial(\frac{u}{p} S_{\max}[\frac{1}{\lambda}]) \subset \frac{u}{p} S_{\max}[\frac{1}{\lambda}]$ implies further that

$$\mathcal{N}^b(y) \in M \otimes_{\mathfrak{S}} \tfrac{u}{p} S_{\max}[\tfrac{1}{\lambda}]$$

Iterating the relation $\mathcal{N}\varphi = p\varphi\mathcal{N}$ yields $\mathcal{N}^b\varphi^n = p^{bn}\varphi^n\mathcal{N}^b$ and as such we find

$$\mathcal{N}^b(x) = p^{bn}\varphi^n(\mathcal{N}^b(y)) \in p^{bn} M^\varphi \otimes \varphi^n(\tfrac{u}{p} S_{\max}[\tfrac{1}{\lambda}])$$

Here we use that $F^0\underline{\mathcal{M}}^\varphi = \underline{\mathcal{M}}^\varphi$ so that $\varphi^n(M) \subset M$. It follows we can write $\mathcal{N}^b(x) = \sum_{k=0}^d \beta_k e_k$ for some $\beta_k \in p^{nb}\varphi^n(\frac{u}{p} S_{\max}[\frac{1}{\lambda}])$. Note that the power series λ is invertible in $W(k)[[\frac{u}{p}]]$ and so $\varphi^n(\frac{u}{p} S_{\max}[\frac{1}{\lambda}]) \subset \varphi^n(\frac{u}{p} W(k)[[\frac{u^p}{p}]])$. Thus we can apply Lemma 9.4.6 to β_k. The result follows. $\square$

LEMMA 9.4.11. *Assume that $F^0\underline{\mathcal{M}}^\varphi = \underline{\mathcal{M}}^\varphi$ and $K_\infty \cap K(\mu_{p^\infty}) = K$. Let $x \in \varphi^n(M)$ be such that $f_\pi(x) \in F^r D_K$. Then for $0 < i \le p^n$ there exists an $x^{(i)}_{\mathrm{trun}} \in F^{\delta_i}\underline{\mathcal{M}}^\varphi$ (recall $\delta_i = \min\{i, r\}$) such that $f_\pi(x^{(i)}_{\mathrm{trun}}) = f_\pi(x)$ and such that*

$$x^{(i)}_{\mathrm{trun}} - x \in p^{\mathcal{L}} M^\varphi$$

where $\mathcal{L} = p^n + n - 1 - (i-1) - v_p((i-1)!)$.

PROOF. Choose an $\mathfrak{S}$-basis $e_1, \dots, e_d$ of M^φ. As a consequence of Lemma 9.4.10 we deduce that there exist $\beta_{k,m,b} \in K_0$ with $v_p(\beta_{k,m,b}) \ge nb + p^n - m - 1 + v_p\binom{p^n}{m}$ such that for $a, b \ge 1$ one has

$$B_{a,b} := \frac{H(u)^a}{a!}\mathcal{N}^b(x) \in \sum_k \sum_{\substack{m \ge 0 \\ a+m<i}} \frac{\beta_{k,m,b}}{\pi^a a!}(u-\pi)^{m+a} e_k + E^i \underline{\mathcal{M}}^\varphi \tag{9.4.12}$$

Note that $E^i\underline{\mathcal{M}}^\varphi \subset F^i\underline{\mathcal{M}}^\varphi$ since $F^0\underline{\mathcal{M}}^\varphi = \underline{\mathcal{M}}^\varphi$. Let $A_{a,b}$ denote the double sum in (9.4.12). Lemma 9.4.8 says that $x^{(i)} - x = \sum z_{a,b} B_{a,b}$ for some $z_{a,b} \in \mathbb{Z}$; set $x^{(i)}_{\mathrm{trun}} = x + \sum z_{a,b} A_{a,b}$. As $i > 0$ we have that $f_\pi(A_{a,b}) = f_\pi(B_{a,b})$. This and the fact that by construction $f_\pi(x^{(i)}) = f_\pi(x)$, we have $f_\pi(x^{(i)}_{\mathrm{trun}}) = f_\pi(x)$. Further, Lemma 9.4.5 tells us that $x^{(i)}_{\mathrm{trun}} \in F^{\delta_i}\underline{\mathcal{M}}^\varphi$. We finish the proof by showing that each $A_{a,b}$ lies in $p^{\mathcal{L}} M^\varphi$. To see this observe that

$$v_p\left(\frac{\beta_{k,m,b}}{\pi^a a!}\right) \ge nb + p^n - (m+a) - 1 + v_p\binom{p^n}{m} - v_p(a!)$$

and that, allowing a, b and m to vary subject to $a, b \geq 1$, $m \geq 0$ and $m + a < i$, the minimal value of the right hand side of this inequality is $\mathcal{L}$ (obtained when $m = 0$, $a = i - 1$ and $b = 1$). $\square$

Recall the map $f_\pi : M^\varphi \to M_{f_\pi}$ of filtered modules defined in Lemma 9.3.3, where $M_{f_\pi} \subset D_{\mathrm{crys}}(V)$ is given the subspace filtration. If T is a crystalline $\mathcal{O}$-lattice then f_π is $\mathcal{O}$-linear and so f_π restricts to give maps of filtered modules $f_\pi : M^\varphi_\tau \to M_{f_\pi,\tau}$ for each $\tau : k \to k_E$.

LEMMA 9.4.13. *Assume that T is a crystalline $\mathcal{O}$-lattice. Let $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, k_E)$ and let $n \geq 1$ be an integer. If $\mathrm{HT}_{\tau\circ\varphi^i}(V) = \{0, \dots, 0\}$ for $0 < i < n$ (when $n = 1$ this is a vacuous condition) then the inclusion $\varphi^n(M_{\tau\circ\varphi^n}) \subset M_\tau$ induces an equality*

$$f_\pi(\varphi^n(M_{\tau\circ\varphi^n})) = M_{f_\pi,\tau}$$

PROOF. Since $\mathrm{HT}_{\tau\circ\varphi^i}(V) = \{0, \dots, 0\}$ for $0 < i < n$ the analogue of Lemma 9.3.5 with coefficients implies that $M^\varphi_{\tau\circ\varphi^i} = M_{\tau\circ\varphi^i}$. Thus, arguing by induction, it follows that $\varphi^n(M_{\tau\circ\varphi^n})$ generates M^φ_τ over $\mathfrak{S}_\tau = \mathcal{O}[[u]]$. Since the map f_π is just reduction modulo E, it follows that $f_\pi(\varphi^n(M_{\tau\circ\varphi^n}))$ generates $M_{f_\pi,\tau}$ over $\mathcal{O}[[u]]/E = \mathcal{O}$. This implies $f_\pi(\varphi^n(M_{\tau\circ\varphi^n})) = f_\pi(M^\varphi_\tau)$ since the latter subset is stable under the action of $\mathcal{O}$ (recall that φ on M is $\mathcal{O}$-linear). $\square$

We are now ready to prove our generalisation of Theorem 4.5.1.

THEOREM 9.4.14. *Suppose that T is a crystalline $\mathcal{O}$-lattice and that $K_\infty \cap K(\mu_{p^\infty}) = K$. For each $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k, k_E)$ choose integers $n_\tau \geq 1$. Assume that for each τ*

$$\mathrm{HT}_{\tau\circ\varphi^i}(V) = \{0, \dots, 0\}$$

for each $0 < i < n_\tau$ and that

$$\mathrm{HT}_\tau(V) \subset [0, p^{n_\tau} - x_{n_\tau}]$$

where for any $n \geq 1$, x_n denotes the smallest integer satisfying $n + x_n > v_p((p^n - x_n - 1)!)$. Then $\overline{M} = M(T)/\varpi$ satisfies the equivalent condition of Lemma 4.4.4, and $\mathrm{Weight}_\tau(\overline{M}) = \mathrm{HT}_\tau(T)$ *for each τ.*

PROOF. First let us show that f_π is strict. This will imply $\mathrm{Weight}_\tau(\overline{M}) = \mathrm{HT}_\tau(T)$ (Corollary 9.3.7). Since all the Hodge–Tate weights of T are ≥ 0 we have $F^0\underline{\mathcal{M}}^\varphi = \underline{\mathcal{M}}^\varphi$. Since all the τ-th Hodge–Tate weights are $\leq p^{n_\tau} - x_{n_\tau}$, $F^r M_{f_\pi,\tau} = 0$ for $r > p^{n_\tau} - x_{n_\tau}$. Therefore to prove f_π is strict we have to show that for any $\overline{x} \in F^r M_{f_\pi,\tau}$ with $r \leq p^{n_\tau} - x_{n_\tau}$ there exists $x_{\mathrm{trun}} \in F^r M^\varphi_\tau$ with $f_\pi(x_{\mathrm{trun}}) = \overline{x}$. By Lemma 9.4.13 there is an $x \in \varphi^{n_\tau}(M_{\tau\circ\varphi^n})$ such that $f_\pi(x) = \overline{x}$. By definition of x_{n_τ} we have that $p^{n_\tau} + n_\tau - 1 - (i-1) - v_p((i-1)!) > 0$ when $i = p^{n_\tau} - x_{n_\tau}$, and so Lemma 9.4.11 provides us with an $x_{\mathrm{trun}} = x^{(i)}_{\mathrm{trun}} \in F^r \underline{\mathcal{M}}^\varphi_\tau$ such that

$$x_{\mathrm{trun}} - x \in pM^\varphi_\tau$$

Thus $x_{\mathrm{trun}} \in M_\tau^\varphi \cap F^r \underline{\mathcal{M}}_\tau^\varphi = F^r M_\tau^\varphi$ (this last equality follows from (1) of Lemma 9.3.3). This proves $\underline{f_\pi}$ is strict.

It remains to prove that $\overline{M}$ satisfies the equivalent conditions of Lemma 4.4.4. Since $f_\pi : M^\varphi \to M_{f_\pi}$ is strict we have

$$\mathrm{gr}^i(M_\tau^\varphi/E) = \mathrm{gr}^i(M_{f_\pi,\tau})$$

The filtration on $M_{f_\pi,\tau}$ is defined so that each $\mathrm{gr}^i(M_{f_\pi,\tau})$ is $\mathcal{O}$-torsionfree. Lemma 4.1.8 may then be applied with $M = M_\tau$, $N = M_\tau^\varphi$ and $A = \mathcal{O}[[u]], a = E$; thus there is an $\mathcal{O}[[u]]$-basis $(\mathfrak{m}_i)$ of M_τ such that $(u^{r_i}\mathfrak{m}_i)$ is a $\mathcal{O}[[u]]$-basis of M_τ^φ. The $u^{r_i}\mathfrak{m}_i$ are obtained by choosing any lifting of an $\mathcal{O}$-basis of $\mathrm{gr}^{r_i}(M_\tau^\varphi/E)$ to $F^{r_i}M_\tau^\varphi$. By the previous paragraph we see that any element of $\mathrm{gr}^{r_i}(M_\tau^\varphi/E)$ can be lifted to an element $x_{\mathrm{trun}} \in F^{r_i}M_\tau^\varphi$ with $x_{\mathrm{trun}} \equiv x$ modulo pM_τ^φ for some $x \in \varphi^{n_\tau}(M_{\tau\circ\varphi^{n_\tau}}) \subset \varphi(M_{\tau\circ\varphi})$. Thus we can arrange that there are $x_i \in \varphi^{n_\tau}(M_{\tau\circ\varphi}) \subset \varphi(M_{\tau\circ\varphi})$ such that $u^{r_i}\mathfrak{m}_i \equiv x_i$ modulo pM_τ^φ. The image (m_i) of the $(\mathfrak{m}_i)$ is then a basis of $\overline{M}$ as in Lemma 4.4.4. This finishes the proof. $\square$

REMARK 9.4.15. Given integers n_τ one could define a full subcategory $\mathrm{Mod}_k^{\mathrm{SD},n_\tau}(\mathcal{O})$ of $\mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ as follows: an object $M \in \mathrm{Mod}_k^{\mathrm{BK}}(\mathcal{O})$ is contained in $\mathrm{Mod}_k^{\mathrm{SD},n_\tau}(\mathcal{O})$ if M satisfies the equivalent conditions of Lemma 4.4.4 and for each τ

$$\mathrm{Weight}_\tau(M) \subset [0, p^{n_\tau}]$$

and $\mathrm{Weight}_{\tau\circ\varphi^i}(N) = \{0,\dots,0\}$ for each $0 < i < n_\tau$. It is straightforward (but somewhat more tedious since almost every argument requires a induction on the n_τ) to show that each result we have proved for $\mathrm{Mod}_k^{\mathrm{SD}}(\mathcal{O})$ holds analogously $\mathrm{Mod}_k^{\mathrm{SD},n_\tau}(\mathcal{O})$.

As a consequence Theorem 7.3.1 holds for crystalline representations as in Theorem 9.4.14. Note that the Hodge–Tate weights allowed by Theorem 9.4.14 are more restrictive than the weights allowed in the definition of $\mathrm{Mod}_k^{\mathrm{SD},n_\tau}(\mathcal{O})$. We believe that Theorem 9.4.14 holds more generally with $x_{n_\tau} = 0$ but we do not know how to prove this.

CHAPTER 10

Serre Weights

In this chapter we define the various terms which appear in Theorem B, and show how to deduce this theorem from Theorem 7.3.1. This is essentially trivial once the required definitions and results from [**18**] and [**1**] have been recalled.

1. Local Serre Weights

In this section we specialise results from [**18**] and [**19**] to the case of GL_n. Let $T_{\mathbb{Z}} \subset \mathrm{GL}_{n,\mathbb{Z}}$ denote the split maximal torus consisting of diagonal matrices. The character group $X = X(T_{\mathbb{Z}})$ can be identified with $\bigoplus_{i=1}^n \mathbb{Z}\epsilon_i = \mathbb{Z}^n$; the character

$$\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \mapsto \prod_i t_i^{a_i}$$

corresponding to $\sum a_i\epsilon_i = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. The set of roots of GL_n are then $R = \{\epsilon_i - \epsilon_j \mid i \neq j\}$ and we choose simple roots $\epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$. The corresponding Borel B_n (respectively the opposite Borel B_n^-) is the group over $\mathbb{Z}$ of upper triangular matrices (respectively the group of lower triangular matrices). We define

$$\mathbb{Z}^n_+ = \{(a_1, \ldots, a_n) \mid a_i \geq a_{i+1}\}$$

so that under the identification $X = \mathbb{Z}^n$ the dominant characters X_+ correspond to $\mathbb{Z}^n_+$. We also set

$$X_s = \{(a_1, \ldots, a_n) \in X \mid 0 \leq a_i - a_{i+1} \leq p^s - 1\}$$

whenever $s \geq 0$. Thus $X_0 = (1, \ldots, 1)\mathbb{Z}$. We can view any $\lambda \in X_+ = \mathbb{Z}^n_+$ as a character of B_n^- via the surjection $B_n^- \to T$.

Definition 10.1.1. Let A be a ring and $\lambda \in X_+$. Define

$$H^0_A(\lambda) = \mathrm{Ind}_{B^-_{n,A}}^{\mathrm{GL}_{n,A}}(\lambda)$$

to be the algebraically induced module. See [**19**, Section II, 8.6].

If A is a field of characteristic 0 then $H^0_A(\lambda)$ is the irreducible representation of highest weight λ. The same is not true in characteristic p.

Definition 10.1.2. Let E be an algebraic extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. For $\lambda \in X_+ = \mathbb{Z}^n_+$ we define

- M_λ be the representation obtained by evaluating $H^0_{\mathbb{Z}}(\lambda)$ on $\mathrm{GL}_n(\mathcal{O})$.

- P_λ be the representation obtained by evaluating $H^0_{\mathbb{F}}(\lambda)$ on $\mathrm{GL}_n(\mathbb{F})$.
- Let F_λ be the sub-$\mathbb{F}$-representation of P_λ generated by the highest weight vector.

These representations coincide with the similarly named representations defined in [**1**, Section 2]. Note however that $H^0_A(\lambda)$ is induced over the opposite Borel whereas in *loc. cit.* the inductions occur over the Borel of upper triangular matrices; this is the reason for the slight differences between the two definitions of M_λ.

There is the following result of Jantzen, see [**18**, Corollary 3.17].

Theorem 10.1.3 (Jantzen). *Suppose that $\mathbb{F}$ is a finite field of cardinality $q = p^f$.*

(1) *For $\lambda \in X_f$ the F_λ are irreducible $\mathbb{F}$-representations of $\mathrm{GL}_n(\mathbb{F})$ and every such irreducible representation arises in this way.*
(2) *We have $F_\lambda \cong F_\mu$ if and only if $\lambda - \mu \in (q-1)X_0$.*
(3) *Fix an embedding $\tau_0 : \mathbb{F} \to \overline{\mathbb{F}}_p$ through which we identify $\mathbb{F} = \mathbb{F}_q$. Every irreducible $\overline{\mathbb{F}}_p$-representation of $\mathrm{GL}_n(\mathbb{F})$ can be written as*

$$\bigotimes_{\tau \in \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}, \overline{\mathbb{F}}_p)} F_{\lambda_\tau} \otimes_{\mathbb{F},\tau} \overline{\mathbb{F}}_p$$

with $\lambda_\tau \in X_1$. Two such tensor products are isomorphic if and only if $\sum_{i=0}^{f-1} (\lambda_{\tau_0 \circ \varphi^i} - \lambda'_{\tau_0 \circ \varphi^i}) p^i \in (q-1)X_0$.

In general P_λ will not be irreducible. However when λ is in the closure of the *lowest alcove* the representations P_λ and F_λ coincide (see [**18**, Proposition 3.16]). In particular we have

Lemma 10.1.4. *Let $\lambda \in X_+ = \mathbb{Z}^n_+$ be such that*

$$\lambda_1 + (n-1) - \lambda_n \leq p$$

Then $F_\lambda = P_\lambda$.

Definition 10.1.5. Let $\mathbb{F}$ be a finite field of characteristic p. A Serre weight for $\mathrm{GL}_n(\mathbb{F})$ is an isomorphism class of irreducible $\mathbb{F}$-representations of $\mathrm{GL}_n(\mathbb{F})$.

Note that the set of Serre weights is in bijection with the set of isomorphism classes of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_n(\mathbb{F})$.

2. The Global Setup

Everything in this section is taken from [**1**, Section 2.1], specialised to the case in which p is unramified in F. Our aim is to globalise the notion of a Serre weight, and explain what it means for a residual representation $\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ to be modular of such a weight, as in Theorem B. Thus let F be an imaginary CM field, with maximal totally real subfield F^+ and assume as in Theorem B that

- F/F^+ is unramified at all finite places.
- Every place $v|p$ of F^+ splits in F.
- If n is even then $n[F^+ : \mathbb{Q}]/2$ is even.
- $p > 2$ is unramified in F.

With these assumptions there exists a reductive algebraic group G/F^+ which is an outer form of GL_n with $G_{/F} \cong \mathrm{GL}_{n/F}$, and such that if v is a finite place of F^+ then G is quasi-split at v, while if v is an infinite place of F^+ then $G(F_v^+) \cong U_n(\mathbb{R})$. Further a model of G over $\mathcal{O}_{F^+}$ may be defined such that if v is a place of F^+ which splits in F as ww^c then there exist isomorphisms $\iota_w : G(\mathcal{O}_{F_v^+}) \cong \mathrm{GL}_n(\mathcal{O}_{F_w})$.

Since p is unramified in F we can identify $\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}_p) = \coprod_{w|p} \mathrm{Hom}_{\mathbb{F}_p}(k_w, \overline{\mathbb{F}}_p)$ where the union runs over the places w of F above p. The global setting we shall consider puts some restriction upon the weights which can appear, and for this reason we define

$$(\mathbb{Z}_+^n)_0^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)}$$

to be the subset consisting of $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)}$ satisfying

$$\lambda_{\tau,i} = -\lambda_{\tau\circ c, n+1-i}$$

Here c denotes complex conjugation. Let us also write

$$(X_1)_0^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)} = X_1^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)} \cap (\mathbb{Z}_+^n)_0^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)}$$

There is a clear relation between these weights and Serre weights, as we have defined, coming from (3) of Theorem 10.1.3. In [**1**] elements of $(X_1)_0^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)}$ are referred to as Serre weights.

Now we summarise the different senses in which a continuous irreducible $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ or $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is modular or automorphic, as described in [**1**, Section 2].

- A cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ is RACSDC (regular, algebraic, conjugate self dual, and cuspidal) if π_∞ has the same infinitesimal character as some irreducible representation of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_n$, and $\pi^c = \pi^\vee$. Such a π is said to have level prime to p if π_v is unramified for all $v|p$. If $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)}$ write Σ_λ for the irreducible algebraic representation of $\mathrm{GL}_n^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)}$ given by the tensor product of τ of the irreducible representation of GL_n of highest weight λ_τ. An RACSDC representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ is said to have weight λ if π_∞ has the same infinitesimal character as $\Sigma_\lambda^\vee$. In this case we necessarily have that $\lambda \in (\mathbb{Z}_+^n)_0^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)}$.

  Fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$. Attached to any RACSDC representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ there is a continuous semisimple representation $r_{p,\iota}(\pi) : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ satisfying a number of properties (see [**1**, Theorem 2.1.2]). Here we record only the property that if π has level prime to p then $r_{p,\iota}(\pi)$ is crystalline and if π has weight λ then for each $\tau : F \to \overline{\mathbb{Q}}_p$ we have

$$\mathrm{HT}_\tau(r_{p,\iota}(\pi)) = \{\lambda_{\tau,1} + n - 1, \ldots, \lambda_{\tau,n}\}$$

We say that a continuous irreducible $r : G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is automorphic if there exists an RACSDC π such that $r \cong r_{p,\iota}(\pi)$, and say r is automorphic of weight $\lambda \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)}$ if π has weight λ. Likewise a continuous irreducible $\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is automorphic (of weight λ) if $\overline{r} \cong \overline{r}_{p,\iota}(\pi)$.

Definition 10.2.1. For each place v of F^+ over p, fix a place $\tilde{v}$ of F above v. Let $\tilde{S}$ denote the set of $\tilde{v}$. With notation as in Definition 10.1.2 take $E = \overline{\mathbb{Q}}_p$, so we obtain representations M_λ and F_λ of the groups $\mathrm{GL}_n(\overline{\mathbb{Z}}_p)$ and $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$ respectively. Recall $\mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}_p) = \coprod_{w|p} \mathrm{Hom}_{\mathbb{F}_p}(k_w, \overline{\mathbb{F}}_p)$, so we can write

$$\lambda = (\lambda_w)_{w|p} \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}_{\mathbb{Q}_p}(F,\overline{\mathbb{Q}}_p)}$$

Define

$$M_\lambda = \bigotimes_{\tilde{v} \in \tilde{S}} M_{\lambda_{\tilde{v}}}$$

which is a representation of $\prod_{\tilde{v}\in\tilde{S}} \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$. We view M_λ as a representation of $G(\mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ via the isomorphisms $\iota_{\tilde{v}} : G(\mathcal{O}_{F^+_v}) \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}_{F_{\tilde{v}}})$. We also define

$$F_\lambda = \bigotimes_{\tilde{v} \in \tilde{S}} F_{\lambda_{\tilde{v}}}$$

which is a representation of $\prod_{\tilde{v}\in\tilde{S}} \mathrm{GL}_n(k_{\tilde{v}})$. If $\lambda \in (X_1)_0^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)}$ then F_λ is irreducible. As with M_λ we can view F_λ as a representation of $G(\mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$. The representations F_λ and M_λ of $G(\mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ do not depend upon the choices of $\tilde{v}$ made.

- Now we describe the notion of being modular in the sense of coming from an algebraic modular form on G. Let $U \subset G(\mathbb{A}^\infty_{F^+})$ be a good compact open subgroup as described in [**1**, Definition 2.1.5] and W a $\overline{\mathbb{Z}}_p$-module with an action of $G(\mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$. We write $S(U, W)$ for the space of algebraic modular forms of level U and weight W, i.e. the space of functions

$$f : G(F^+)\backslash G(\mathbb{A}^\infty_{F^+}) \to F_\lambda$$

satisfying $f(gu) = u_p^{-1} \cdot f(g)$ for all $u \in U$. Here u_p denotes the image of u under the projection from $G(\mathbb{A}^\infty_{F^+})$ onto $G(\mathcal{O}_{F^+} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$. Let T be a finite set of finite places of F^+ which split in F, containing every place above p and every place v which splits in F and is such that $U_v \neq G(\mathcal{O}_{F^+_v})$. Define the Hecke algebra $\mathbb{T}^{T,\mathrm{univ}}$ to be the commutative $\overline{\mathbb{Z}}_p$-algebra generated by formal variables $T_w^{(j)}$

for all $1 \leq j \leq n$ and all places w of F lying above a place v of F^+ which splits in F and is not contained in T. The algebra $\mathbb{T}^{T,\mathrm{univ}}$ acts on $S(U,W)$ via the Hecke operators

$$T_w^{(j)} = \iota_w^{-1}\left[\mathrm{GL}_n(\mathcal{O}_{F_w})\begin{pmatrix}\varpi_w I_j & 0\\ 0 & I_{n-j}\end{pmatrix}\mathrm{GL}_n(\mathcal{O}_{F_w})\right]$$

where ϖ_w denotes a uniformiser of F_w. If $\mathfrak{m}$ is maximal ideal of $\mathbb{T}^{T,\mathrm{univ}}$ with residue field $\overline{\mathbb{F}}_p$ and is such that $S(U,M_\lambda)[\frac{1}{p}]_{\mathfrak{m}} \neq 0$ then there exists a continuous semisimple representation

$$\overline{r}_{\mathfrak{m}} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$$

which is uniquely determined by a list of properties given in [**1**, Section 2]. We say that a continuous irreducible $\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is modular of weight $\lambda \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)}$ if there exists a good level U which is sufficiently small (i.e. there exists a finite place v of F^+ the projection from U to $G(F_v^+)$ contains no elements of finite order other than the identity) and a maximal ideal $\mathfrak{m}$ of $\mathbb{T}^{T,\mathrm{univ}}$ (for some finite set of places T) such that

- $S(U,F_\lambda)_{\mathfrak{m}} \neq 0$ (this implies $S(U,M_\lambda)[\frac{1}{p}]_{\mathfrak{m}} \neq 0$ so the representation $\overline{r}_{\mathfrak{m}}$ exists), and
- $\overline{r} \cong \overline{r}_{\mathfrak{m}}$.

We say a continuous representation $\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ has split ramification if $\overline{r}|_{G_{F_w}}$ is unramified for any finite place w of F which does not split in F^+. If $\overline{r}$ is modular of some weight then it must have split ramification.

We conclude this section by stating the following lemma which illustrates the relation between the notions of automorphy and modularity given above.

LEMMA 10.2.2. *Let $\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be a continuous irreducible representation with split ramification. If $\overline{r}$ is modular of weight λ there is an RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight λ and level prime to p, which is unramified at all finite places of F which do not split over F^+, such that $\overline{r} \cong \overline{r}_{p,\iota}(\pi)$. Conversely, if there is an RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight λ and level prime to p which is unramified at all finite places of F which do not split over F^+, and is such that $\overline{r} \cong \overline{r}_{p,\iota}(\pi)$, then $\overline{r}$ is modular of some weight $\mu \in (\mathbb{Z}^n_+)_0^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)}$ such that the representation P_λ has F_μ as a Jordan–Holder factor.*

This is [**1**, Lemma 2.1.11].

3. Proof of Theorem B

Maintain the notation from the previous section. In the following definition we globalise the notions of $W(\overline{\rho})^{\mathrm{crys}}$ and $W(\overline{\rho})^{\mathrm{diag}}$ defined in the introduction.

DEFINITION 10.3.1. Let $\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be a continuous representation satisfying $\overline{r}^c = \overline{r}^\vee \chi_{\mathrm{cyc}}^{1-n}$. We let $W(\overline{r})^{\mathrm{crys}}$ be the set of $\lambda = (\lambda_w) \in (X_1)_0^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)}$ such that for each $w \mid p$ the local representation $\overline{r}|_{G_{F_w}}$ has a crystalline lift whose τ-th Hodge–Tate weights for each $\tau \in \mathrm{Hom}_{\mathbb{F}_p}(k_w, \overline{\mathbb{F}}_p)$ are

$$\{\lambda_{w,\tau,1} + n - 1, \ldots, \lambda_{w,\tau,n}\}$$

Likewise we define $W(\overline{r})^{\mathrm{diag}}$ to be those weights for which $\overline{r}|_{G_{F_w}}$ has a potentially diagonalisable crystalline lift. We write $W(\overline{r})^{\mathrm{diag}}_{\leq p}$ and $W(\overline{r})^{\mathrm{crys}}_{\leq p}$ for the subsets consisting of λ with

$$\lambda_{w,\tau,1} + n - 1 - \lambda_{w,\tau,n} \leq p$$

THEOREM 10.3.2. *Suppose $\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is irreducible with split ramification. Assume that*

- *there is an RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that*
  - *$\overline{r} \cong \overline{r}_{p,\iota}(\pi)$ (so that in particular $\overline{r}^c \cong \overline{r}^\vee \chi_{\mathrm{cyc}}^{1-n}$).*
  - *For each place $w \mid p$ of F, $r_{p,\iota}(\pi)|_{G_{F_w}}$ is potentially diagonalisable.*
  - *$\overline{r}(G_{F(\zeta_p)})$ is adequate.*

Let $\lambda \in (X_1)_0^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)}$ and assume that $\lambda \in W(\overline{r})^{\mathrm{diag}}$. Then there is a $\mu \in (X_1)_0^{\mathrm{Hom}_{\mathbb{Q}}(F,\overline{\mathbb{Q}}_p)}$ such that

- *$\overline{r}$ is modular of weight μ.*
- *There is a Jordan–Holder factor of P_λ which is isomorphic to F_μ.*

This is [**1**, Theorem 4.1.9]. Using this we can prove Theorem B.

PROOF OF THEOREM B. Let $\lambda \in W(\overline{r})^{\mathrm{crys}}_{\leq p}$. Then $P_\lambda = F_\lambda$ (Lemma 10.1.4) and so, by the above theorem, it suffices to show $\lambda \in W(\overline{r})^{\mathrm{diag}}_{\leq p}$. This will follow if for each place $w \mid p$ above $W(\overline{r}|_{G_{F_w}})^{\mathrm{diag}}_{\leq p} = W(\overline{r}|_{G_{F_w}})^{\mathrm{crys}}_{\leq p}$. If $\overline{r}_{G_{F_w}}$ is semisimple this follows from Theorem A. If $\overline{r}|_{G_{F_w}}$ satisfies any of the other conditions from Theorem B this follows from Corollary 8.2.3. $\square$

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